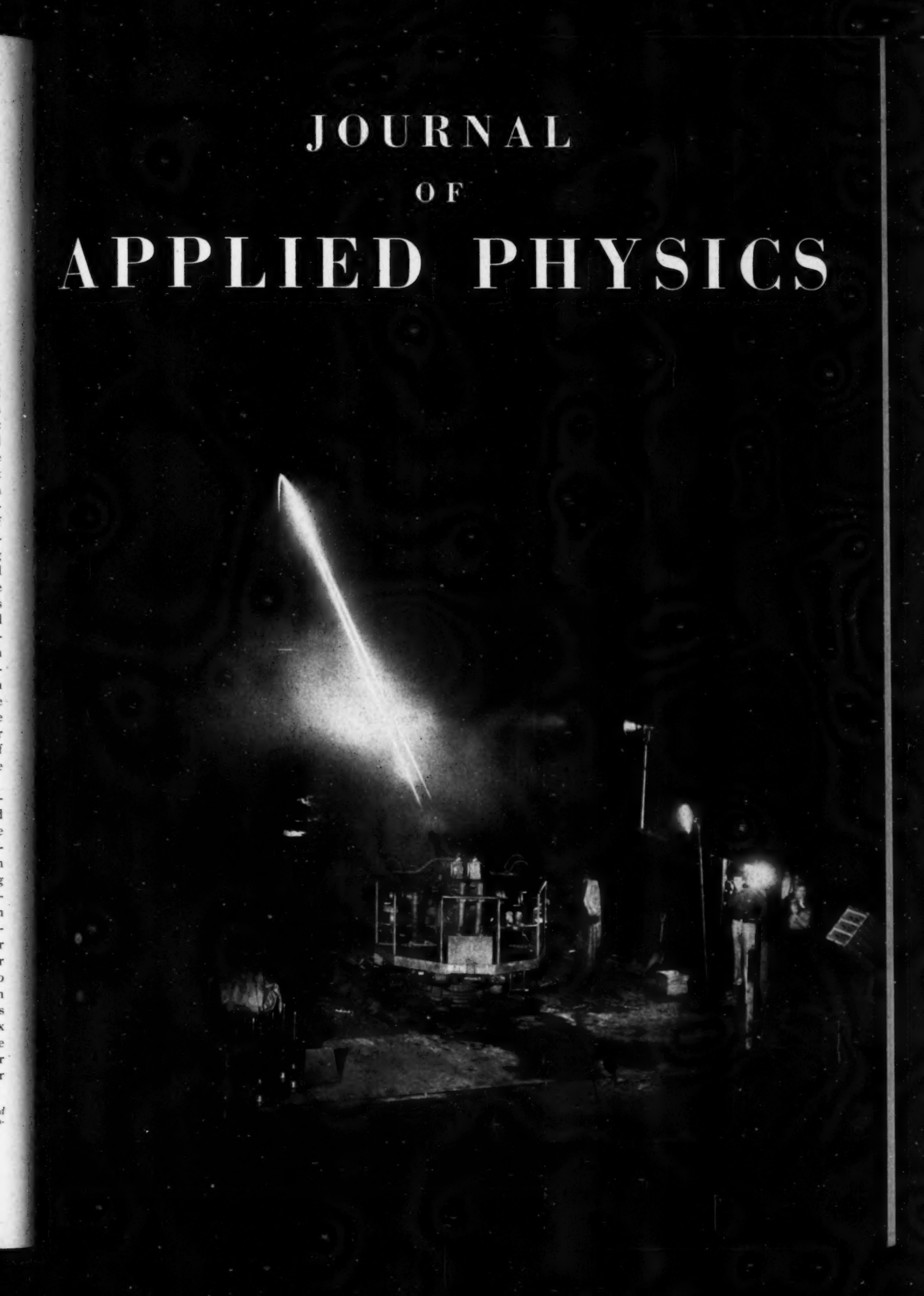
JOURNAL OF APPLIED PHYSICS



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Volume 15, Number 7 July, 1944

Concerning the Division of Electron and Ion Optics of the American Physical Society

BY J. R. PIERCE

Secretary-Treasurer, Division of Electron and Ion Optics

PRIOR to the inaugural meeting of the new Division of Electron and Ion Optics, held at Pittsburgh on April 29 of this year, the members of the Division were invited to express their particular interests in the field and to seek and to contribute to clarification of the aims of the Division. I believe that the questions raised may be of interest to non-members as well as present members of the Division. Several general questions were raised and these perhaps deserve first attention.

Generally, the aims and purposes of the Division are stated in the by-laws to be "the advancement of knowledge of electron and ion optics. To achieve this, the Division shall arrange for the presentation, at meetings of the Society or meetings of its own, of papers and discussions lying within its scope." Such activities may well be supplemented by encouragement of publication and other dissemination of information through correspondence and other activity on the part of the membership and officers. The Division plans no special publications. When meetings are held concurrently with general meetings of the American Physical Society, abstracts will appear in the Bulletin and be reprinted in *The Physical Review*.

The scope of the Division is such as to include both theoretical and practical aspects of production and utilization of ion and molecular beams. It should certainly include consideration of both low frequency and high frequency aspects of electron flow, whether that flow occurs in a wholly new device, a mass spectrograph, or a vacuum tube. Many members of the Division will be interested in new developments and techniques in the use of the electron microscope. However, the particular field which no other organization seems to include so directly is the internal physics of electronic and ionic devices.

One member wants to know whether the aims and purposes of the Division are beneficial to the parent society. I certainly hope so. Another is concerned with the social import of this branch of physics, of which I can say little. One is interested in management of research and development, a very concrete and vitally important problem of no little difficulty.

Turning to specific technical matters, by far the greatest number of replies indicated a primary interest in what might be termed general electron and ion optics. Interest was expressed in the primarily mathematical aspects of electron optics and in mathematics from the point of view of design, in electric and magnetic lenses and focusing means, in relativistic electron optics, especially in connection with magnetic analyzers for beta-particle spectra, in electron and ion

sources, in beams with low energy spread, and in molecular beams.

More specifically, the electron microscope and electron diffraction were very frequently mentioned. Other members expressed particular interest in the mass spectrograph, the cyclotron, x-ray tubes, and the electron optics of radio tubes.

One member showed a particular interest in oxide-coated cathodes. I do not know whether this puzzling subject, together with the general study of thermionic emission, is naturally within the scope of the Division; I believe that it well may be. It is certainly a field in which much remains to be done.

As examples of how the activities of the Division may be of help to the members, I mention that on the program of the Pittsburgh meeting we had invited papers covering several of the subjects in which interest was expressed, including the historical background of the topic of most interest, general electron optics, and papers on the perhaps somewhat neglected optics of vacuum tubes as well as papers on the very popular subject of electron microscopy.

As another example of such service as might be rendered through the Division, I refer to an excellent bibliography of the electron microscope, which was recently prepared in another connection by Claire Marton, the wife of the Vice Chairman of the Division, and Samuel Sass. I think it should be a duty and activity of the Division to encourage the preparation of such bibliographies, of summaries of fields such as have appeared in *Reviews of Modern Physics* and in the *Journal of Applied Physics*, and also of simple tutorial papers which may serve as introductions to the various aspects of electron and ion optics which I have mentioned earlier.

Finally, one member has suggested that after the war a symposium might be held to consider new techniques and data that have not been released during the war. I certainly hope that such a symposium can be held soon. In the meantime, it may be possible to hold at a future meeting a symposium on some less secret but nonetheless vital matter.

New Books

Mathematics of Modern Engineering, Volume II

By Ernest G. Keller. Pp. 309+xii, Figs. 87, 15½×24 cm. John Wiley & Sons, Inc., New York, 1942. Price $4.00.

This book, second in a series of three, considers discrete physical problems which lead to ordinary differential equations. It is stated that the material and methods included have evolved out of research and the advanced course in engineering at the General Electric Company.

The principal purpose of the book is to provide by examples an introduction to the methods of mathematical engineering, this field being described as a new phase of engineering, and defined as those portions of engineering which can be formulated mathematically. As in mathematical physics, the fundamental method of mathematical engineering consists of: (1) the reduction of the physical phenomena involved to a mathematical system amenable to treatment; and (2) the solution of the system. Step one, usually the most difficult, requires originality, insight, and a broad knowledge of the physical and mathematical sciences. Step two, while often difficult, can usually be carried through by known mathematical technique.

The value of this book lies in its collection of problems illustrating the application of mathematics to certain types of engineering problems. The examples relating to locomotives reflect the author's experience and are particularly interesting.

The general theory is treated rather unsystematically. Considerable improvement would have resulted if the fundamental definitions and principles had been introduced accurately and early in the exposition, rather than (as in a number of instances) in the style of an afterthought. Thus: the various definitions of non-holonomic systems (pp. 20–29); the use of Koenig's theorem before its statement (never adequately given, pp. 20–1, 31, 44); the treatment of vector time derivatives given is often inadequate (even for approximations, pp. 45–7). The inclusion of Kron's mysterious "first postulate" seems questionable; the presence of Example I (with its last sentence) immediately thereafter may lead to false notions.

Chapter I, "Engineering dynamics and mechanical vibrations," is the best. Chapter II, ". . . Tensor analysis of stationary networks and rotating electrical machinery," (Kron's theories) is the least convincing part. Chapter III, "Non-linearity in engineering," considers: (1) methods of obtaining, or approximating, solutions to ordinary differential equations; (2) elliptic functions; (3) extensive bibliography.

Richard S. Burington
Washington, D. C.

The National Bureau of Standards

The National Bureau of Standards, Washington, D. C., has announced two new mathematical tables which are now obtainable. They are the following:

Table of Fourier Coefficients (MT23). By Arnold N. Lowan and Jack Laderman. Reprinted from *Journal of Mathematics and Physics*, September, 1943, 11 pages. Obtainable from the National Bureau of Standards, Washington, D. C., for the sum of 25 cents.

Coefficients for Numerical Differentiation with Central Differences (MT23). By Herbert E. Salzer. Reprinted from *Journal of Mathematics and Physics*, September, 1943, 21 pages. Obtainable from the National Bureau of Standards, Washington, D. C., for the sum of 25 cents.

Mathematics of Radio Communications

By T. J. Wang. Pp. 371+ix, Figs. 199, 14×22 cm. D. Van Nostrand Company, Inc., New York, 1943. Price $3.00.

The vast expansion of the radio industry and its associated sister electronic industries, particularly in connection with the war, has stimulated a demand for a large number of technicians trained in the more practical aspects of laboratory work in electronics, but without the broad training of engineers. The principal obstacle in the way of advancement of these radio technicians is usually an inadequate training in mathematics occasioned by the lack of sufficient time to include the necessary formal courses in mathematics in their educational curricula.

To remedy this situation, the book reviewed here sketches an outline for a course in the mathematics of most frequent use to workers in electronics laid out in a manner that may be given concurrently with classroom and laboratory work in electricity. The subject material has been well chosen and includes topics in arithmetic, algebra, analytic geometry, trigonometric functions, complex vectors as applied to circuit analysis, differential and integral calculus, and Fourier series. The examples and problems have been ingeneously chosen from many branches of electrical communication work, including such subjects as a.c. and d.c. circuit analysis, electron trajectories, vacuum-tube characteristics, analysis of oscilloscope figures, and the elements of modulation theory. This, together with a concise but well-written text, will no doubt contribute considerably toward making the book more interesting than the more usual mathematics texts to the group for whom it was written.

Unfortunately, the lack of answers to any of the problems will possibly inhibit the use of the book for self-study by a sizeable group of workers in the field, who no longer have the opportunity to attend classes, but who would have welcomed a volume of this type whose contents they might have mastered at home.

Leslie L. Foldy
Columbia University, Division of War Research

Basic Mathematics for War and Industry

By Paul H. Daus, John M. Gleason, and William M. Whyburn. Pp. 277+xi, Figs. 277, 14×22 cm. The Macmillan Company, New York, 1944. Price $2.00.

On the home front, no less than on the fighting fronts, the present war has pointed out indisputably that the

basic mathematics ordinarily taught in high schools is not a dry academic subject limited in its usefulness to the torment of unfortunate students under the guise of "mental discipline," but that it is instead an active and useful tool for the solution of the most practical problems of modern warfare and industry.

In view of the war-inspired need for rapid training, and review of large classes of our population in basic high school mathematics, the authors of the present book have tried to select the most useful portions of the usual syllabus in these subjects, and judiciously integrate them with their most important present military and industrial applications in a small volume which would prove equally valuable as a textbook in civilian and armed service training courses, as well as for home study. They have included in the volume material on arithmetic, algebra, plane and solid geometry, and plane and spherical trigonometry. There is a literal profusion of worked-out examples and problems (with answers supplied), the majority of which are drawn from machine shop work, air and sea navigation, physics, and similar war important applications. The entire emphasis is on mathematics as a tool, and the choice and organization of the material have been such as to expedite rapid familiarization with applications, rather than appreciation of mathematics for its own sake. Thus, while the conventional form of presentation of geometry as a series of propositions and constructions derived from axioms and postulates is retained, the logical aspects of the subject are completely submerged even to the extent of including considerably more axioms than are logically necessary. The text is lucidly and coherently written, and the typography is large and clear, making an attractive volume.

This book is recommended to all who lacked the opportunity to study high school mathematics or who were unfortunate enough to forget what they had once learned and who now find that such knowledge would be valuable to them in their present military or civilian occupation.

LESLIE L. FOLDY
Columbia University, Division of War Research

Mr. Tompkins Explores the Atom

BY G. GAMOW. Pp. 97+x, 17.5×22 cm. The Macmillan Company, New York, 1944. Price $2.00.

When discoveries have been made in physics by the application of new mathematical techniques to old problems, there usually follow periods during which the full physical significance of the results is poorly apprehended by the majority of scientists. Gradually, while the new developments become more familiar by being discussed and applied in wider fields, their meaning grows more apparent, and simple verbal interpretations are found to supplement the original mathematical ones. For beginners, particularly, some mechanical model is usually the most acceptable, although once an initial period of professional apprenticeship has been served, it is the young men who go on to develop new ideas with confidence and enthusiasm.

During this period of interpretation, it takes a man of unusual skill to create the necessary word pictures even for a technical audience; and one of imagination, like Professor Gamow, to make a translation into the everyday language of the layman. In the latter case, an apparently fantastic story must inevitably follow. Perhaps this is why, in *Mr. Tompkins Explores the Atom*, Professor Gamow has proceeded with his hero's education by giving him painless dreams which reveal the fundamental properties of the primary particles as though they were the natural ingredients of a fairy tale.

In three dreams, which occupy the first half of this little volume, Mr. Tompkins, or his wife, finds himself or herself on easy speaking terms with Maxwell's demon, with the Electron tribe, and with the maker of Nuclei, from all of whom they receive intelligible explanations of the world of atoms. To drive his points home, the author adds in the second half of the book four lectures by "The Professor" who illustrates by skillful analogy the behavior of atoms and their nuclei. A non-scientific reader will enjoy the fairy tales; a physicist, especially if he can recognize something of Rutherford's features in the face of the woodcarver, will chuckle as he scans the other illustrations; and an advanced undergraduate student will find the dreams and the lectures useful as collateral reading during a first course in nuclear structure. Only a careless reader will suffer from philosophic insomnia by reason of the typographical error which introduces "fluctuations of destiny" on p. 10.

If J. M. Barrie had been a scientist, M'Connachie, flying around lightheartedly on one wing, dragging with him a reluctant author who would have preferred to be a family lawyer standing firm on his hearthrug among the harsh realities of office furniture—M'Connachie, instead of Mr. Tompkins, might have been the hero of a story such as this.

THOMAS H. OSGOOD
Michigan State College

Fundamentals of Optical Engineering

BY DONALD H. JACOBS. Pp. 487+x, Figs. 343, 14×21½ cm. McGraw-Hill Book Company, Inc., New York, 1943. Price $5.00.

This is a timely text suitable for a first course in the design of optical instruments. The wartime importance of this subject is brought home by the nature of the instruments discussed. This may cause much of the book to appear overspecialized in postwar times. However, the basic principles may be readily applied to broader fields by one who has mastered them.

The book consists of four parts of which the first, consisting of 151 pages, presents an excellent summary of the principles of geometrical optics. The theory of stops is presented in a manner that makes this important subject easy to grasp by a beginner. This is followed by a discussion of representative instruments such as the various types of telescopes, gunsights, and rangefinders used in the military service.

The third part, dealing with mechanical and electrical design, is a unique feature of this book. Many engineers differ in opinion as to whether optical design should precede

mechanical design or vice versa. The author observes that these are but two aspects of a single problem and should be worked out concurrently. He states that a primary purpose in writing the book is to show how mechanical and optical design are interrelated. Not only is this accomplished, but the student is given a good practical introduction into mechanical principles and machine-shop operations as they apply to optical instruments. It seems strange, however, that optical-shop procedures and machines are not discussed while the mechanical ones are. Then again, optical tolerances are treated, but methods of inspection are not. No doubt the reason for these omissions is that their inclusion would make the book inconveniently large for a text on the fundamentals of design.

The fourth and final section specifically deals with optical design. Ray-tracing methods and some short-cut formulae are presented. The theory is illustrated by application to the design of aplanatic objectives and eyepieces. The treatment is elementary but sound and can serve as an excellent introduction to advanced texts such as that by Conrady, whose notation has been adopted.

The book is profusely illustrated by drawings and some halftones. There are numerous suggestions for further reading at the end of each chapter. There are enough problems to aid the student in testing his knowledge and in gaining mastery through application. This is an excellent textbook on a subject of great importance.

JOSEPH VALASEK
University of Minnesota

Structure of Metals

BY CHARLES S. BARRETT. Pp. 567+xiii, Figs. 388, 15×23 cm. McGraw-Hill Book Company, New York, 1943. Price $6.00.

Dr. Barrett has discussed in one book all the aspects of metals which have been studied by means of x-rays. These aspects include not only those covered in standard crystallographic treatises, but also those developed extensively during the past fifteen years. These latter include order-disorder phenomena, residual stresses, the changes occurring during cold working and the resulting structure and preferred orientations resulting therefrom, the preferred orientations accompanying crystallization from the melt and accompanying recrystallization of deformed metals, the intermediate steps in age hardening and in phase transformation. The related subject of electron diffraction is treated briefly. The discussion of this large range of topics is not confined exclusively to the findings of x-rays. Other pertinent experimental data are also introduced, as well as theoretical interpretations.

It is believed this book is the first in English to describe clearly the method and use of the concept of reciprocal lattice.

Dr. Barrett has had a long and varied experience in the application of x-rays to the solution of problems in the structure of metals. This rich background is reflected in the style with which his book is written. The general physics of each topic is discussed with a minimum of mathematics so that the book may be read fairly rapidly. On the other hand, an exhaustive reference to the literature enables one to find readily more complete discussions of the details.

C. ZENER
Watertown Arsenal

Magnetochemistry

BY PIERCE W. SELWOOD. Pp. 287+ix, Figs. 80, 16×23½ cm. Interscience Publishers, Inc., New York, 1943. Price $5.00.

This book gives a useful survey of the field of magnetochemistry, defined by the author as "the application of magnetic susceptibilities and of closely related quantities to the solution of chemical problems." It covers portions of the field of magnetism different from the well-known books of Van Vleck and Stoner, and is a welcome supplement to them. The latter books are concerned primarily with the relations between the susceptibility and the structure of the atom or molecule, expressed in terms of spin and orbital quantum numbers and temperature. In Selwood's book the author starts with these theoretical formulae and discusses the changes in structure that occur as one goes from one atom to the next in the periodic table or as one combines a given electropositive atom with a variety of electronegative ones, or as the groupings of organic radicals are changed. Magnetic properties are interpreted in terms of atomic structure as expressed by the quantum theory, and magnetic measurements are shown to provide a powerful tool for discriminating between various hypothetical structures.

The field covered by this book is similar to that treated by Klemm in his *Magnetochemie* published in 1936. Since this date over 1000 papers on magnetochemistry have been published and the consideration given to them by Selwood makes his book a competent up-to-date review of the subject.

The plan of the book may be stated briefly. A chapter at the beginning is concerned with methods of measurement and treatment of the data. A chapter at the end on "Applied magnetometric analysis" mentions some applications valuable chiefly for metallurgists. The other seven chapters forming the main body of the book are devoted to the diamagnetism and paramagnetism of atoms, molecules, and metals, and (briefly) to the ferromagnetism of metals and compounds.

An important part of the presentation of data is the free use of tables—showing, for example, the variation of susceptibility with position in the vertical columns of the periodic table—and of structural formulae for organic compounds.

The book seems particularly adapted to the student who is planning to do research in magnetochemistry. It can be read with pleasure and profit by many others concerned with the interesting magnetic manifestations of atomic and mulecular structure, and the copious references to original sources enable the student to extend his reading with a minimum of effort.

R. M. BOZORTH
Bell Telephone Laboratories

Here and There

The Establishment of a Division of High Polymer Physics in the American Physical Society

Dr. Karl K. Darrow, secretary of the American Physical Society, recently announced that a Division of High Polymer Physics had been authorized by the council of the society, acting in response to a petition signed by thirty-one fellows and members.

The field of this division is defined as the advancement and diffusion of knowledge of the physics of high polymeric materials such as rubbers, textiles, and plastics. Meetings of the division may be held (if the council so authorizes) in conjunction with or separately from meetings of the society.

The council has appointed an "organizing committee" to take charge of the organization and the affairs of the division until by-laws are formulated and adopted by the council and a mechanism is set up for electing an executive committee. The organizing committee consists of F. G. Brickwedde, K. K. Darrow, G. B. Pegram, and A. E. Ruark, selected from the council; and R. B. Barnes, W. F. Busse, P. Debye, J. H. Dillon, W. J. Lyons, and L. A. Wood, selected from the members sponsoring the division. W. J. Lyons (Southern Regional Research Laboratory, 2100 Robert E. Lee Boulevard, New Orleans 19, Louisiana) is serving as secretary of this committee.

According to Article IX of the constitution of the society:

1. The council may, upon petition by members of the society, form a division within the society charged with the advancement and diffusion of the knowledge of a specified subject or subjects in physics.

2. Each division shall elect an executive committee, the chairman of which shall report its activities and needs to the council.

3. Any division may be dissolved at the discretion of the council.

National Electronics Conference

Designed to fulfill the need for providing a common meeting place and discussion forum for technical personnel interested in applications in the broad field of electronics, the first annual National Electronics Conference will be held at the Medinah Club of Chicago, 505 North Michigan Avenue, Chicago 11, Illinois, on October 5–7, 1944. Emphasis will be placed on the engineering aspects of electronics in the conference which is described as a "national forum for electronic developments and their engineering applications." The conference is sponsored jointly by the Illinois Institute of Technology, Northwestern University, the Chicago Section of the Institute of Radio Engineers, and the Chicago Section of the American Institute of Electrical Engineers. Technical papers and discussion groups are planned to treat the communication, industrial, measurement, instrumentation, control, scientific, and medical applications of electronics.

Those wishing to submit papers for presentation at the conference are invited to communicate with Professor A. B. Bronwell, Chairman of the Program Committee, Technological Institute, Northwestern University, Evanston, Illinois, or Mr. Beverly Dudley, Secretary of the National Electronics Conference, 520 North Michigan Avenue, Chicago 11, Illinois.

National Science Teachers Association

Formation of the National Science Teachers Association, "to stimulate, improve, and coordinate science teaching at elementary, secondary, and collegiate levels of instruction," was announced by Dr. Philip G. Johnson, President *pro tem* of the new organization. It has been formed as the first step in a merger of two national science teachers' organizations, the American Science Teachers Association and the American Council of Science Teachers. It is expected to represent all teachers of science, and it will be affiliated with the American Association for the Advancement of Science and the National Education Association. Membership in the association will be open to all science teachers, and to anyone interested in science teaching. Provision is made for affiliation of other science teachers' groups, and any such organization may affiliate. Headquarters of the association, for the present, will be at Cornell University.

The Charles L. Mayer Nature of Light Awards

The National Science Fund announces two Charles L. Mayer awards for contributions submitted before January 1, 1946 on the nature of light. A prize of $2000 will be awarded for an outstanding contribution to our basic understanding of the nature of light and other electromagnetic phenomena which provides in terms intelligible to the community of scientists at large a unified understanding of the two aspects of these phenomena which are at present jointly described by wave and by corpuscular theories. The second prize of $2000 will be awarded for an outstanding comprehensive contribution to a logical, consistent theory of the interaction of charged particles with an electromagnetic field including the interaction of particles moving with relatively high speeds.

The first award is intended to encourage for the benefit of the non-specialist the interpretation of facts already known to the specialist. It is hoped that the second award will stimulate attack on one of the most fundamental unsolved problems in physics. In making these awards the National Science Fund will have the assistance of a special Advisory Committee consisting of Dr. E. U. Condon, Associate Director of the Research Laboratories, Westinghouse Electric & Manufacturing Company, East Pittsburgh, Pennsylvania; Dr. Karl K. Darrow, member of the technical staff of the Bell Telephone Laboratories, and Secretary of the American Physical Society; Dr. Robert A. Millikan, Chairman of the Executive Council, California Institute of Technology; and Dr. I. I. Rabi, Professor of Physics, Columbia University.

Contributions may be submitted to the National Science Fund of the National Academy of Sciences, 2101 Constitution Avenue, Washington 25, D. C.

News Notes

Dr. John T. Tate, Dean of the College of Science, Literature, and the Arts of the University of Minnesota, who has been on leave since 1941, has asked to be relieved of administrative responsibilities. On completing his work with the National Defense Research Committee he will return to the university as Research Professor of Physics.

Dr. Willis R. Whitney, honorary Vice President of the General Electric Company and first director of its research laboratory, has been elected an honorary member of the Electrochemical Society. The certificate of honorary membership will be conferred on him at the Milwaukee meeting.

Hendrik Johannes Van Der Bijl, Research Physicist, Director of War Supply, Union of South Africa, Vice Chancellor, Pretoria University, has been elected a fellow of the Royal Society of London.

The University of Wisconsin will confer the degree of doctor of science on Dr. Jesse T. Littleton, physicist, Assistant Director of Research and Development with the Corning Glass Works, Corning, New York.

Earl Russell (Bertrand Russell), who has lived in the United States for some years, has been elected a fellow of Trinity College, Cambridge. He expects to return to England during the summer to resume his work in philosophy and mathematics.

The Electrochemical Society

The Eighty-Fifth Convention of the Electrochemical Society was held April 13–15 in Milwaukee. Dr. R. M. Burns, President of the Society, was in charge. The theme of the convention was "The electrochemist's contribution to the war effort." The main technical session was on "powder metallurgy."

Honorary membership in the Society was bestowed on Paul J. Kruesi, President of Southern Ferro Alloys Company, Chattanooga, Tennessee, and Willis R. Whitney of the General Electric Company, Schenectady, New York. The Young Author's Cash Prize and Book Prize go to Walter G. Berl of Carnegie Institute of Technology.

The Physics Club of Philadelphia

At a recent meeting of the Physics Club of Philadelphia held at the University of Pennsylvania, Dr. Martin A. Pomerantz of the Bartol Research Foundation gave an address on "The nature of cosmic-ray particles."

Necrology

Dr. Margaret E. Maltby, Associate Professor of Physics, retired, of Barnard College, Columbia University, died on May 3 at the age of eighty-three years.

Dr. Lionel Robert Wilberforce, Professor of Physics at the University of Liverpool from 1900 until his retirement in 1935 with the title emeritus, died on April 1 at the age of eighty-two years.

Erratum

Through a regrettable accident, part of the heading in the advertisement of Ballantine Laboratories on page ii of the May issue was dropped out in the printing. It should have read 0.00002 to 10,000 volts.

New Booklets

The Harshaw Chemical Company has released a booklet entitled *Synthetic Optical Crystals* which describes the development of the large scale growing process and the production of synthetic crystals. A limited number of booklets are offered free upon request by The Harshaw Chemical Company, 1945 East 97 Street, Cleveland, Ohio.

Nickel Steel Topics, Vol. 13, No. 1, March, 1944, is an 8-page booklet which is published in the interest of producers and users of nickel alloy steel alloys. Requests for copies should be addressed to The International Nickel Company, Inc., 67 Wall Street, New York 5, New York, attention of H. S. Lewis, Editor.

Chemical Digest, Vol. 10, No. 1, Spring, 1944, contains an article entitled "Chemical products for postwar needs" as well as other items concerning chemical news. Copies of this 4-page publication are available upon request from Foster D. Snell, Inc., 305 Washington Street, Brooklyn 1, New York, attention F. E. Parry.

Wheelco Comments, Vol. 3, No. 3, January-February, 1944, is a 4-page booklet which presents articles on the problems of industrial measurement and control. It is published by Wheelco Instruments Company, Wheelco Building, Harrison and Peoria Streets, Chicago 7, Illinois, and available on request

A new folder which describes the new model Spencer refractometer is available and will be supplied by the Spencer Lens Company, 19 Doat Street, Buffalo 11, New York.

The new *Cannon Electric Laboratory and Switchboard Connector Bulletin* contains information, photographs, drawings, and data on this line of specialty fittings. Catalogued are surface and sub-mounting plugs and receptacles, straight cord plugs and receptacles, switching plugs, and experimental switchboards. This 12-page bulletin will be sent upon request from Cannon Electric Development Company, 3209 Humboldt Street, Los Angeles 31, California.

The *Ohmite News*, April, 1944, contained a résumé of the career of Heinrich Friedrich Emil Lenz originator of Lenz's law. Described also is the Ohmite rheostat and its applications. Ohmite Manufacturing Company, 4835–41 Flournoy Street, Chicago, Illinois is the publisher (2 pages).

Contributed Original Research

Transmission Line Theory Applied to Wave Guides and Cavity Resonators. I*

DAVID MIDDLETON AND RONOLD KING
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(Received February 1, 1944)

In Part I circuits consisting of hollow metal wave guides of arbitrary cross section are analyzed in terms of scalar and vector potential and stream functions in a form representing a generalization of the treatment of the conventional lines given by R. King. Completely hyperbolic solutions analogous to those given in the usual theory are obtained for the E and H modes; these yield general relations for the stream and potential distributions. Terminal, input, and characteristic impedances are defined for the wave guide or cavity resonator. The propagation constant is discussed, and it is demonstrated that the attenuation (constant) due to the imperfect metal walls of the pipe and the dielectrics in the guide are additive if the materials have low loss. The characteristic impedances for the two types of mode are considered, and general formulae for the practical conditions of operation are included. The analogy between this theory and that of the conventional line is discussed, and it is pointed out that the physical interpretations of results formally identical in the two cases are equivalent for a given mode and a particular cross section of pipe. The analogy breaks down, however, when one attempts to treat different modes and guides of different transverse dimensions at the same time, for example, in the problem of impedance matching. One advantage of the theory is that experimentally meaningful results may be obtained for general terminations which ordinarily cannot be handled analytically.

INTRODUCTION

THE general analysis of the conventional open-wire and coaxial transmission lines, closely spaced compared with a wave-length, is well known, and has been expressed in many different forms. It has also been shown that the fundamental differential equations relating the total current I and the potential difference V for the conventional line are formally analogous to the relation governing the behavior of metal pipes and coaxial tubes whose cross sections are sufficiently large to allow higher modes in the distribution of charge and current. This formal analogy is obtained by replacing the current I and the voltage difference V of the conventional line with a scalar (or vector) stream and potential function, respectively, from which it is possible to determine the fields and the charge and current densities.

Perhaps the most convenient general solution of the transmission line equations is the completely hyperbolic one, given by R. King,[1] in which the various terminal impedances are expressed in terms of hyperbolic functions. The main advantage of this form is that the over-all effects of the attenuation and phase shift of the terminations may be added, respectively, and directly, to the over-all attenuation αs and phase shift βs of the line itself, and that the solutions are quite general, and in the practical cases yield simple results for the determination of such quantities as input impedance, standing wave ratios, transfer of power, etc.

It is our purpose in this paper, which at least in Part I is to be regarded as a companion piece to reference 1, to show in detail that the analysis of transmission circuits constructed of hollow and coaxial conductors may be given in the general hyperbolic form, mentioned above, of the ordinary theory, and to apply that theory in a number of practical cases. Specifically we will define and discuss terminal and input impedances of a wave guide and cavity resonator

* A considerable part of Part I is a modification of course lectures given by the second-named writer at Harvard University during the last three years.

[1] R. King, J. App. Phys. **14**, 577 (1943).

of arbitrary cross section, unrestricted terminations, and driven at some arbitrary point along the tube. A brief discussion of the propagation constant and characteristic impedances will be included, and finally in Part II we will consider a number of experimental examples, and a theoretical treatment of a closed, cylindrical pipe containing several imperfect dielectric media at one end. The great advantage of the analysis lies in the fact that definite and meaningful experimental results may be obtained, for *any* termination, whereas a mathematical treatment of the problem would, in most cases, be quite impractical if not impossible.

Since, as will be shown, there exists a formal analogy between our results and those of the conventional theory, the formulae obtained in one case apply equally well, formally, in the other, so that it is a comparatively simple matter to obtain results in either system, once the necessary electrical and geometrical constants of the particular transmission circuit under examination have been measured. For this reason we will attempt to follow the notation of a recent article on transmission lines,[1] and will on occasion quote formulae rather freely from it, to speed the use of our derivations in practical problems. It must be emphasized, however, that the analogy between the two cases is not complete, and hence care must be taken in the interpretation of the results obtained in one case when applied in the other. We will consider the nature of the analogy in a more detailed manner at the end of Part I.

PART I. DEVELOPMENT OF THE THEORY

In deriving expressions for the stream and potential distributions, from which we will make our definitions of impedance, we proceed first from Maxwell's equations for media free of sources and sinks:

$$\nabla\times\mathbf{E}=-i\omega\mu\mathbf{H}=-Z_m\mathbf{H};$$

$$Z_m\equiv i\omega\mu,\ (\text{ohms/m}), \quad (1)$$

$$Y_e\equiv i\omega\epsilon^*,\ (\text{ohm-m})^{-1}.$$

$$\nabla\times\mathbf{H}=i\omega\epsilon^*\mathbf{E}=Y_e\mathbf{E}; \quad (2)$$

$$\epsilon^*\equiv\epsilon-i\sigma/\omega.$$

$$\nabla\cdot\mathbf{H}=0, \quad (3)$$

$$\nabla\cdot\mathbf{E}=0. \quad (4)$$

These relations are the fundamental field equations for the steady state, or forced oscillation phenomena such as we are dealing with here, for which the time variation of the fields is of the form $e^{i\omega t}$, where ω is the angular frequency of the source. No restriction is placed on the various media other than that they be simple, homogeneous, and isotropic, the usual conditions in practice. Then the (absolute) permittivity ϵ, permeability μ, conductivity σ, measured in fárads/m, henrys/m, and mhos/m, respectively, are constants independent of the applied and induced fields. Because of the solenoidality of the **E** and **H** fields, *vide* Eqs. (3), (4), vector stream and potential functions may be defined, along with a pair of scalar functions which are chosen in such a way that the vector quantities are uniquely specified. That this may be done follows at once from the definition of a uniquely determined vector,[2] which requires that its curl and divergence be known, and that it vanish at infinity as $1/r^2$. As we shall see presently, the pair of functions derived from $\nabla\cdot\mathbf{E}=0$ gives rise to transverse electric (TE),[3] or H-type modes, and those from $\nabla\cdot\mathbf{H}=0$ yield the transverse magnetic (TM)[3] or E-type modes, where the E and H types are distinguished here from one another by the superscripts $q=1, 2$, respectively.

In order to have available results to which we shall wish to refer later let us proceed in a somewhat more detailed fashion than we would ordinarily. Considering the H modes first, we define a vector potential $\mathbf{A}^{(2)}$ in volts from (4) as follows:

$$\nabla\times\mathbf{A}^{(2)}\equiv-\mathbf{E}^{(2)}, \quad (5)$$

since the operator $\nabla\cdot\nabla\times$ always yields zero. From (2) this gives

$$\nabla\times(\mathbf{H}^{(2)}+Y_e\mathbf{A}^{(2)})=0, \quad (6)$$

from which a scalar stream function $\Psi^{(2)}$ (amperes) may also be defined, so that

$$\nabla\Psi^{(2)}\equiv-(\mathbf{H}^{(2)}+Y_e\mathbf{A}^{(2)}), \quad (7)$$

whereupon we obtain

$$\mathbf{H}^{(2)}=-Y_e\mathbf{A}^{(2)}-\nabla\Psi^{(2)}. \quad (8)$$

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[2] M. Mason and W. Weaver, *The Electromagnetic Field* (University of Chicago Press, 1929), theorem IV, page 357.

[3] S. A. Schelkunoff, *Electromagnetic Waves* (D. Van Nostrand Company, Inc., New York, 1943), pp. 153, 154.

Now

$$\nabla\times\nabla\times\mathbf{A}^{(2)}=-\nabla\times\mathbf{E}^{(2)}=Z_m\mathbf{H}^{(2)}$$
$$=-Z_m(Y_e\mathbf{A}^{(2)}+\nabla\Psi^{(2)}). \quad (9)$$

In order to be able to separate the variables and to determine $\mathbf{A}^{(2)}$ uniquely, we relate $\Psi^{(2)}$ and $\mathbf{A}^{(2)}$ by the following equation

$$\nabla\cdot\mathbf{A}^{(2)}=-Z_m\Psi^{(2)}. \quad (10)$$

From (9) and (10) and the vector identity $\nabla\cdot\nabla\mathbf{A}=\nabla\nabla\cdot\mathbf{A}-\nabla\times\nabla\times\mathbf{A}$ we obtain finally the familiar vector and scalar wave equations for the transverse electric modes:

$$\nabla\cdot\nabla\mathbf{A}^{(q)}+k^2\mathbf{A}^{(q)}=0, \quad (11)$$

$$\nabla^2\Psi^{(q)}+k^2\Psi^{(q)}=0, \quad q=1, 2, \quad (12)$$

where

$$k^2\equiv -Z_mY_e=\omega^2\mu\epsilon^*. \quad (13)$$

In a similar way, because of the vanishing of the divergence of $\mathbf{H}$, we may define for the E modes a vector stream function $\mathbf{A}^{(1)}$ (amperes) and a scalar potential $\Psi^{(1)}$ (volts). The uniqueness condition (10) now becomes

$$\nabla\cdot\mathbf{A}^{(1)}=-Y_e\Psi^{(1)}, \quad (14)$$

and Eqs. (11) and (12) are seen to hold equally well in this case. The fields are given by

$$\mathbf{E}^{(1)}=-Z_m\mathbf{A}^{(1)}-\nabla\Psi^{(1)}, \quad (15)$$

$$\mathbf{H}^{(1)}\equiv\nabla\times\mathbf{A}^{(1)}. \quad (16)$$

The vectors $\mathbf{A}^{(q)}$, $q=1, 2$, may in general have three components; however, this is a considerable and unnecessary complication, as the vector operation $\nabla\cdot\nabla$ does not admit the solution of Eq. (11) by the usual method of separation of variables for non-Cartesian coordinate systems. This difficulty may be avoided if we specify that $\mathbf{A}^{(q)}$ has only one component. No generality is lost; the fields of a particular mode are still completely determined. The most general expression of the fields requires a superposition of both the E ($q=1$) and H ($q=2$) types of oscillation.[4] This requirement is true, regardless of the number of components of $\mathbf{A}^{(q)}$. For axially symmetric systems, such as the ones considered in this paper, the natural choice of the components of $\mathbf{A}^{(q)}$ is obviously that in the z direction, i.e.,

$$\mathbf{A}^{(q)}=\mathbf{i}_3A_g^{(q)}, \quad q=1, 2, \quad (17)$$

where $\mathbf{i}_3$ is a unit vector in the z direction. With this in mind, and separating the variables, we may write (11) and (12) as

$$\nabla^2_{u,v}A_g^{(q)}+(\gamma^{(q)})^2A_g^{(q)}=0, \quad (18)$$

$$\frac{\partial^2A_g^{(q)}}{\partial z^2}-(K_g^{(q)})^2A_g^{(q)}=0, \quad q=1, 2, \quad (19)$$

and similar relations for $\Psi^{(q)}$. Here $\nabla^2_{u,v}$ is the Laplacian in the curvelinear coordinates u, v of the tube cross section. The quantity $\gamma^{(q)}$ is the "shape factor," and is (in general) complex if the walls of the wave guide are not perfectly conducting; otherwise $\gamma^{(q)}$ is real. From the separation of variables we obtain the following relation between $K_g^{(q)}$ and $\gamma^{(q)}$:

$$(K_g^{(q)})^2=(\gamma^{(q)})^2-k^2=[\alpha_g^{(q)}+i\beta_g^{(q)}]^2, \quad (20)$$

and as we shall see shortly $K_g^{(q)}$ may be identified as the propagation constant of the pipe. If we use (10), (14), and (17), and similar relations for the TM modes we find at once the familiar equations

$$\frac{\partial A_g^{(q)}}{\partial z}=-\begin{pmatrix}Y_e\\Z_m\end{pmatrix}\Psi^{(q)}, \quad (21)$$

and

$$\frac{\partial\Psi^{(q)}}{\partial z}=-[K_g^{(q)}]^2A_g^{(q)}/\begin{pmatrix}Y_e\\Z_m\end{pmatrix}, \quad (22)$$

where the upper quantity in braces applies for $q=1$ and the lower for $q=2$. These equations are seen to be formally identical with those of the conventional line,[5] provided we *define* characteristic impedances for the E and H modes by

$$Z_{ce}\equiv K_g^{(1)}/Y_e, \quad (23)$$

$$Z_{cm}\equiv Z_m/K_g^{(2)}, \quad (24)$$

and provided we identify the potential and stream functions of a particular mode with the corresponding potential difference V and the total current I of the usual theory. (See Fig. 1.)

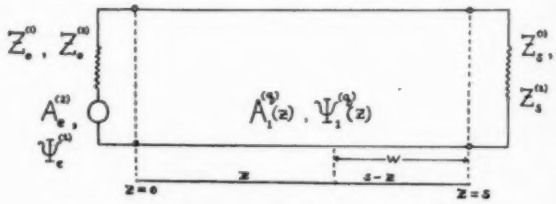

FIG. 1. Circuit diagram of a wave guide and cavity resonator.

[4] E. Ledinneg, Ann. d. Physik **41**, 557 (1942).

[5] See reference 3, Chapter VI, Section 11.

SOLUTIONS OF THE AXIAL EQUATIONS AND DEFINITION OF THE TERMINAL AND INPUT IMPEDANCES

II

Typical solutions of the axial equations (19) are*

$$A_1^{(q)}(u,v)e^{i\omega t}=\left[C_1^{(q)}\begin{Bmatrix}\cosh K^{(1)}z\\ \sinh K^{(2)}z\end{Bmatrix}+D_1^{(q)}\begin{Bmatrix}\sinh K^{(1)}z\\ \cosh K^{(2)}z\end{Bmatrix}\right]T^{(q)}(u,v)e^{i\omega t}, \quad (25)$$

where the upper and lower lines apply when $q=1, 2$, respectively. The functions $T^{(q)}(u,v)$ are solutions of the cross-sectional equations (18) and in the next section will be considered briefly for two special cases. Relations similar to (25) may be obtained for $\Psi^{(q)}$ at once from (21). The constants $C_1^{(q)}$ and $D_1^{(q)}$ will now be determined from the boundary conditions at $z=0, s$. Considering the H modes first we find that the boundary conditions on (19) are, at $z=0$,

$$A_1^{(2)}(0)=A_0^{(2)}(\text{volts}) \quad (26)$$

for which $A_0^{(2)}$ is not specifically known in general, and is a function of the source, detector, and any other apparatus present in the pipe. If $A_e^{(2)}$ (volts) represents the potential distribution of the source for a given mode we have

$$A_e^{(2)}=A_e^{(2)}(u,v)=B_e^{(2)}T^{(2)}(u,v), \quad (27)$$

and then, from Fig. 1, we see that

$$A_0^{(2)}\equiv A_e^{(2)}-\Psi_0^{(2)}Z_0^{(2)}, \text{ (volts)}, z=0. \quad (28)$$

This defines the terminal impedance $Z_0^{(2)}$, which, like $A_0^{(2)}$ and $\Psi_0^{(2)}$, depends on the source, its coupling to the guide, etc. From (10), or what is the same thing in view of (17), from (21), we obtain for (28)

$$A_0^{(2)}=A_e^{(2)}+\frac{Z_0}{Z_m}\frac{\partial A_1^{(2)}}{\partial z}\bigg|_{z=0}, \quad (29)$$

where we specify in general that

$$A_0^{(2)}\equiv B_0^{(2)}T^{(2)}(u,v); \quad (30)$$

here $B_0^{(2)}$ is a constant depending on the amplitude of the source, and Eq. (30) implies that our termination at $z=0$ is sufficiently distant, a wave-length or so in practice, from the exciting probe (antenna) so that the transverse distribution of the vector and stream functions is that of the natural mode excited, i.e., is the solution of (18). We will not attempt to consider the near-zone development of the fields, which is at best an extremely laborious problem, for the far-zone theory treated here is in most cases of much greater practical importance.

For the arbitrary termination at $z=s$ we *define* the impedance by

$$Z_s^{(2)}\equiv A_1^{(2)}(s)/\Psi_1^{(2)}(s)=-Z_m A_1^{(2)}(s)\Big/\frac{\partial A_1}{\partial z}\bigg|_{z=s} \text{ (ohms)}. \quad (31)$$

By satisfying the appropriate conditions for $\mathbf{E}^{(2)}$ and $\mathbf{H}^{(2)}$ at the boundary $z=s$, which in turn requires that we can determine the electromagnetic field distributions for the termination, we obtain an explicit analytical expression for $Z_s^{(2)}$. Usually this is not possible; but it will be demonstrated in Part II for the special case of a resonator or guide terminated in several transverse sections of dielectric material.

In an analogous fashion we find for the E modes that

$$\Psi_0^{(1)}\equiv Y_e^{(1)}-Z_0^{(1)}A_0^{(1)} \text{ (volts)}, (z=0). \quad (32)$$

This defines the terminal impedance $Z_0^{(1)}$, similar to (28). From (14) it follows immediately that

$$\Psi_e^{(1)}-Z_0^{(1)}A_0^{(1)}=-Z_e\frac{\partial A_1}{\partial z}\bigg|_{z=0}; \quad Z_e=1/Y_e. \quad (33)$$

Also we have

$$A_1^{(1)}(0)=A_0^{(1)}\equiv B_0^{(1)}T^{(1)}(u,v),$$

and

$$\Psi_e^{(1)}(u,v)\equiv B_e^{(1)}T^{(1)}(u,v). \quad (34)$$

At $z=s$ the terminal impedance for the TM modes is defined as

$$Z_s^{(1)}\equiv\Psi_1^{(1)}(s)/A_1^{(1)}(s)=-Z_e\frac{\partial A_1^{(1)}}{\partial z}\bigg|_{z=s}\Big/A_1^{(1)}(s) \text{ (ohms)}. \quad (35)$$

With these definitions, (28), (31), (32), (35), as one would expect, our problem becomes formally analogous to that of the conventional line, schematically represented in Fig. 1, with the A's

* We have omitted the subscripts g on A, Ψ, K, etc., whenever there is no ambiguity in so doing.

and Ψ's, depending on the mode excited, replaced by the appropriate I's and V's of the usual theory. The final results are then expressible in the completely hyperbolic form,[1] and we have for the H modes

$$\begin{Bmatrix} \Psi_1^{(2)} \text{ (amp.)} \\ A_1^{(2)} \text{ (volts)} \end{Bmatrix} = \frac{A_s^{(2)}(u,v)\sinh\theta_0^{(2)}}{\sinh(K^{(2)}s+\theta_0^{(2)}+\theta_s^{(2)})} \begin{Bmatrix} 1/Z_{cm}\sinh(K^{(2)}w+\theta_s^{(2)}) \\ \cosh(K^{(2)}w+\theta_s^{(2)}) \end{Bmatrix}, \quad (36)$$

where the upper line in the braces applies for $\Psi_1^{(2)}$, and the lower for $A_1^{(2)}$. For the E modes we obtain similarly

$$\begin{Bmatrix} A_1^{(1)} \text{ (amp.)} \\ \Psi_1^{(1)} \text{ (volts)} \end{Bmatrix} = \frac{\Psi_s^{(1)}(u,v)\sinh\theta_0^{(1)}}{\sinh(K^{(1)}s+\theta_0^{(1)}+\theta_s^{(1)})} \times \begin{Bmatrix} 1/Z_{ce}\cosh(K^{(1)}w+\theta_s^{(1)}) \\ \sinh(K^{(1)}w+\theta_s^{(1)}) \end{Bmatrix}. \quad (37)$$

Here w is the distance from the load or termination $Z_s^{(q)}$ to the point z where $A_1^{(q)}$ and $\Psi_1^{(q)}$ are computed. Thus

$$w = s - z. \quad (38)$$

The time variation may be simply introduced by multiplying (36) and (37) through by $e^{i\omega t}$. The complex quantities $\theta_0^{(q)}$, $\theta_s^{(q)}$ are defined in terms of the ρ and Φ functions[1] by

$$\theta_0^{(q)} \equiv \rho_0^{(q)} + i\Phi_0^{(q)}, \quad \theta_s^{(q)} \equiv \rho_s(q) + i\Phi_s^{(q)}, \quad (39)$$

and the terminal impedances are related to ρ and Φ in the following manner:

$$Z_0^{(q)} \equiv \begin{Bmatrix} Z_{ce} \\ Z_{cm} \end{Bmatrix} \coth\theta_0^{(q)}; \qquad Z_s^{(q)} \equiv \begin{Bmatrix} Z_{ce} \\ Z_{cm} \end{Bmatrix} \coth\theta_s^{(q)}. \quad (40)$$

If $\rho_s^{(q)}$ and $\Phi_s^{(q)}$ are known, for example, from experimental data, we may obtain the terminal resistance and reactance at $z=s$ from

$$R_s^{(q)} = R_c\{{}^{e}_{m}\}\left[\frac{\sinh 2\rho_s^{(q)} - \phi_c\{{}^{e}_{m}\}\sin 2\Phi_s^{(q)}}{\cosh 2\rho_s^{(q)} - \cos 2\Phi_s^{(q)}}\right], \quad (41)$$

and

$$X_s^{(q)} = -R_c\{{}^{e}_{m}\} \times \left[\frac{\sin 2\Phi_s^{(q)} + \phi_c\{{}^{e}_{m}\}\sinh 2\rho_s^{(q)}}{\cosh 2\rho_s^{(q)} - \cos 2\Phi_s^{(q)}}\right]. \quad (42)$$

Here the upper line in the subscripts $\{{}^{e}_{m}\}$ is to be taken for the E modes and the lower line for the H. Reciprocal relations for $\rho_s^{(q)}$ and $\Phi_s^{(q)}$ in terms of $R_s^{(q)}$ and $X_s^{(q)}$ may be found at once from (41) and (42). We have then

$$\rho_s^{(q)} = \tfrac{1}{2}\tanh^{-1} \times \left[\frac{2R_c\{{}^{e}_{m}\}(R_s^{(q)} - \phi_c\{{}^{e}_{m}\}X_s^{(q)})}{R_s^{(q)2} + X_s^{(q)2} + R_c^2\{{}^{e}_{m}\}(1+\phi_c\{{}^{e}_{m}\}^2)}\right], \quad (43)$$

and *

$$\Phi_s^{(q)} = \tfrac{1}{2}\tan^{-1} \left[\frac{-2R_c\{{}^{e}_{m}\}(X_s^{(q)} + \phi_c\{{}^{e}_{m}\}R_s^{(q)})}{R_s^{(q)2} + X_s^{(q)2} - R_c^2\{{}^{e}_{m}\}(1+\phi_c\{{}^{e}_{m}\}^2)}\right]. \quad (44)$$

$\Phi_s \doteq \pi/2$ for a metal piston.

It is often convenient to define the dimensionless ratios r_1 and x_1 by writing

$$Z_s^{(q)}/Z_c\{{}^{e}_{m}\} \equiv r_{1s}^{(q)} + ix_{1s}^{(q)}, \quad (45)$$

where, following King,[6] we have

$$r_{1s}^{(q)} = \frac{R_s^{(q)} - \phi_c\{{}^{e}_{m}\}X_s^{(q)}}{R_c\{{}^{e}_{m}\}(1+\phi_c\{{}^{e}_{m}\}^2)} = \frac{\sinh 2\rho_s^{(q)}}{\cosh 2\rho_s^{(q)} - \cos 2\Phi_s^{(q)}}, \quad (46)$$

and

$$X_{1s}^{(q)} = \frac{X_s^{(q)} + \phi_c\{{}^{e}_{m}\}R_s^{(q)}}{R_c\{{}^{e}_{m}\}(1+\phi_c\{{}^{e}_{m}\}^2)} = \frac{-\sin 2\Phi_s^{(q)}}{\cosh 2\rho_s^{(q)} - \cos 2\Phi_s^{(q)}}. \quad (47)$$

In all but a few critical cases the terms involving $\phi_c\{{}^{e}_{m}\}$ are negligible for good guides, and so we have

$$r_{1s}^{(q)} \doteq R_s^{(q)}/R_c\{{}^{e}_{m}\}; \qquad x_{1s}^{(q)} \doteq X_s^{(q)}/R_c\{{}^{e}_{m}\}. \quad (48)$$

The definitions (40) may be extended immediately to include the input impedances in the following way:

$$Z_{in}^{(q)} \equiv \begin{Bmatrix} Z_{ce} \\ Z_{cm} \end{Bmatrix} \coth(A_s^{(q)} + iF_s^{(q)}), \quad (49)$$

* Here and in reference 1 the following convention is used: Φ_s is in quandrant I if the signs of the argument of the arctangent in (44) are $+/+$; in quadrant II if the signs are $+/-$; in quadrant III if the signs are $-/+$; in quadrant IV if the signs are $+/-$.

[6] Reference 1, formulas (19a) and (19b).

where

$$A_s^{(q)}=\alpha^{(q)}s+\rho_s^{(q)}; \quad F_s^{(q)}=\beta^{(q)}s+\Phi_s^{(q)}. \quad (50)$$

Then we may obtain the input resistance and reactance for the E and H modes from (41) and (42) simply by replacing the subscript s by in and $\rho_s^{(q)}$, $\Phi_s^{(q)}$ by $A_s^{(q)}$ and $F_s^{(q)}$, respectively, and similarly for the reciprocal relations (43) and (44).

The expressions developed by King,[1] formulas (21)–(71), including Table I and Figs. 6, 7, 12 of that paper, may be applied equally well in the corresponding analysis of wave guides given here, so we will not attempt to carry the details of the discussion of impedances and admittances further.

POTENTIAL AND STREAM DISTRIBUTIONS

III

Before examining the axial behavior of $A_1^{(q)}(z)$ and $\Psi_1^{(q)}(z)$, let us consider the transverse equations (18). Particular solutions may be obtained at once for the coordinate systems for which the variables (u, v) are separable.[7] For the two particular cases of a rectangular and circular cross section, respectively, we have the following possible solutions, from which we may construct the proper one to fit the transverse boundary conditions below:

$$T^{(q)}(u,v)=\begin{Bmatrix}\sin a_m^{(q)}x\\ \cos a_m^{(q)}x\end{Bmatrix}\begin{Bmatrix}\cos b_n^{(q)}y\\ \sin b_n^{(q)}y\end{Bmatrix} \quad \begin{matrix}m=0,1,2,3,\cdots\\ n=0,1,2,3,\cdots\end{matrix} \quad (51)$$

$$(a_m^{(q)})^2+(b_n^{(q)})^2=(\gamma_{m,n}^{(q)})^2; \quad 0\leqq x\leqq a; \quad 0\leqq y\leqq b;$$

and for the circular pipe of radius a

$$T^{(q)}(u,v)=J_n(\gamma_{np}^{(q)}r)\begin{Bmatrix}\cos n\theta\\ \sin n\theta\end{Bmatrix}, \quad \begin{matrix}n=0,1,2,3,\cdots\\ p=1,2,3,\cdots\end{matrix} \quad (52)$$

$$0\leqq r\leqq a; \quad 0\leqq\theta\leqq 2\pi.$$

Here J_n is a Bessel function of the first kind and order n. The functions listed in (51), (52) are sufficient for our discussion and for the analysis of the problem treated in Part II. [For a more complete list of solutions to (18) in the circular case, the reader is referred to any standard work on differential equations.]

The boundary conditions to be satisfied at the inner walls of the hollow pipe are the familiar ones[8] requiring, for homogeneous and isotropic media, that the tangential component of the **E** fields and normal component of the **H** fields vanish, i.e.,

$$\mathbf{n}\times\mathbf{E}_1=0; \quad \mathbf{n}\cdot\mathbf{H}_1=0; \quad \mathbf{D}_1=\epsilon\mathbf{E}_1; \quad \mathbf{B}_1=\mu\mathbf{H}_1. \quad (53)$$

If η is the surface charge density (coulombs/m) and **I** the surface current density (amperes/m), then for perfectly conducting walls η and **I** are determined from

$$\mathbf{n}\cdot\mathbf{D}_1=\eta; \quad \mathbf{n}\times\mathbf{H}_1=\mathbf{I}. \quad (54)$$

The volume density of current is $\mathbf{J}=\sigma\mathbf{E}$ (amp./m^2), and exists only in the dielectric, provided, of course, that $\sigma_{\text{dielectric}}\neq 0$. The subscripts (1) on (53) and (54) denote the interior of the hollow pipe. If the walls are not perfectly conducting, however, the fields do not vanish at the inner surfaces of the tube or within region (2), and (53) and (54) become

$$\mathbf{n}\times(\mathbf{E}_1-\mathbf{E}_2)=0; \quad \mathbf{n}\cdot(\mathbf{B}_1-\mathbf{B}_2)=0, \quad (55)$$

$$\mathbf{n}\cdot(\mathbf{D}_1-\mathbf{D}_2)=\eta; \quad \mathbf{n}\times(\mathbf{H}_1-\mathbf{H}_2)=0. \quad (56)$$

Using the asymmetrical form of Green's theorem applied to a bounded surface,[9,10] and with the aid of (18), we obtain the following relation for the "shape factor" $\gamma^{(q)}$ of the hollow pipe:

$$(\gamma^{(q)})^2=\frac{\displaystyle\int\int_S[\nabla_{u,v}T^{(q)*}\cdot\nabla_{u,v}T^{(q)}]dS-\int_s T^{(q)*}(\mathbf{n}\cdot\nabla T^{(q)})ds}{\displaystyle\int\int_S T^{(q)}T^{(q)*}dS}, \quad (57)$$

where S is the cross section of the wave guide and s its periphery. For perfectly conducting walls $\mathbf{n}\cdot\nabla T^{(q)}$ or $T^{(q)*}$ vanish, as a consequence of (53), and $(\gamma^{(q)})^2$ is seen to be real. However,

[7] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), p. 349.

[8] Reference 7, pp. 34–38.

[9] Reference 3, Chapter 10, Section 4.

[10] R. I. Sarbacher and W. A. Edson, *Hyper and Ultrahigh Frequency Engineering* (John Wiley & Sons, Inc., New York, 1943), Chapter VI, Sec. 19, Chapter VII, Sec. 14.

if the walls are not perfectly conducting, the line integral in (57) does not become zero, because of (55), and $(\gamma^{(q)})^2$ is then complex.

The general problem of hollow pipe transmission may be considered in two parts, one, in which the source is closely coupled to the wave guide, or (in some cases) to a cavity resonator, and the other, in which the source is loosely coupled, approaching the free oscillation condition. For the transmission of power we are primarily interested in close coupling and in having the generator at one end of the system over which energy is being propagated. The solution of this problem has already been obtained in the previous section, and the final equations of the potential and stream distributions are given by (36) and (37). Since we have strong coupling we cannot speak of "point" sources, in the conventional transmission line case, or correspondingly, of "plane" sources for the hollow pipe, but rather we must specify a volume source distribution, whose exact nature it is rarely, if ever, within our power to describe accurately. For this reason we have not attempted to analyze the fields, charge, current, and potential distributions in the immediate neighborhood of the source, but have assumed that at a certain distance from it, and any other apparatus present, detectors, etc., of no more than about a wave-length in the closely coupled cases, and somewhat less for loose coupling, the transverse field pattern of the particular mode excited is known, and corresponds to that of a natural mode of the tube. This procedure eliminates the intricate and approximate analysis which attempts to deal with source geometries, and is reasonably accurate even if the generator does not have the same field pattern in its near zone as it does at a distance of about a wave-length. It is important to note (and it will be considered more fully at the end of Part I) that for the wave guide or resonator theory the potential and stream distributions are three-dimensional, comprising a cross-sectional as well as longitudinal variation, whereas the current and potential difference are one-dimensional in the conventional transmission line analysis.

It is possible to generalize the solution obtained in the previous section to account for a volume source distribution at any region along the tube. This is not the problem of greatest practical interest, however, since generally for power transfer it is more usual to have the source at one end, feeding a load at the other, a problem already successfully treated.

For the case of loose coupling we can consider the generator to be a plane source, whose field distribution is known over a cross section of infinitesimal width. Such a distribution may be physically realized by a series of antennae suitably arranged. A single antenna will also serve, but we cannot assume that the relations for the fields, charges, etc. will hold to within the same distance of the source as for the more elaborate distribution.

The loosely coupled case may then be easily generalized for a source at any point along the tube.[11] If the guide (or resonator) is driven by a loosely coupled oscillator, which sets up a cosine-symmetrical field with respect to the planes $z=x_s$[12,13] along the pipe, then it is sufficient to replace $\theta_0^{(q)}$ in the numerator of (36) and (37) by $\theta_0^{(q)}+K^{(q)}x_s$ and to write $s-x_s$ for s, with s the new over-all length.

We then have

$$\begin{Bmatrix}\Psi_1^{(2)}\\ A_1^{(2)}\end{Bmatrix}=\frac{A_{e,x_s}^{(2)}(u,v)\sinh(K^{(2)}x_s+\theta_0^{(2)})}{\sinh(K^{(2)}s+\theta_0^{(2)}+\theta_s^{(2)})}\times\begin{Bmatrix}1/Z_{cm}\sinh(K^{(2)}w+\theta_s^{(2)})\\ \cosh(K^{(2)}w+\theta_s^{(2)})\end{Bmatrix},\quad(58)$$

for the H modes, and for the E modes

$$\begin{Bmatrix}A_1^{(1)}\\ \Psi_1^{(1)}\end{Bmatrix}=\frac{\Psi_{e,x_s}^{(1)}(u,v)\sinh(K^{(1)}x_s+\theta_0^{(1)})}{\sinh(K^{(1)}s+\theta_0^{(1)}+\theta_s^{(1)})}\times\begin{Bmatrix}1/Z_{ce}\cosh(K^{(1)}w+\theta_s^{(1)})\\ \sinh(K^{(1)}w+\theta_s^{(1)})\end{Bmatrix}.\quad(59)$$

If the loosely coupled oscillator sets up a sine-symmetrical field, so that it is equivalent to a "double layer" plane distribution of sources, symmetrically placed with respect to x_s, we have results similar to (58) and (59), except that

$$\sinh(K^{(q)}x_s+\theta_0^{(q)})$$

[11] For the conventional line see R. King, Proc. I. R. E. **27**, 715 (1939).
[12] See also R. King, Proc. I. R. E. **28**, 84–87 (1940).
[13] R. King, Rev. Sci. Inst. **10**, 325 (1939).

is replaced by

$$\cosh\,(K^{(q)}x_s+\theta_0^{(q)}),$$

$$A_{e,x_s}^{(2)} \text{ and } \Psi_{e,x_s}^{(1)} \text{ by } \bar{A}_{e,x_s}^{(2)} \text{ and } \bar{\Psi}_{e,x_s}^{(1)}.$$

For an asymmetrical oscillator a sum of distributions using $A_{e,x_s}^{(2)}$ and $\bar{A}_{e,x_s}^{(2)}$ for the H modes and $\Psi_{e,x_s}^{(1)}$ and $\bar{\Psi}_{e,x_s}^{(1)}$ for the E modes may be employed. Cosine-symmetrical distributions may be obtained by placing the antenna, in the form of a circular loop at a distance of n half-wave-lengths in the guide from one end of the tube. When a straight wire antenna is located an odd number of quarter-wave-lengths from one end, sine-symmetrical fields will result.[14–17]

In practice we are usually concerned with the magnitude of $A_1^{(q)}$ and $\Psi_1^{(q)}$, which become for the case of a source at one end, from (36) and (37):

$$\left|\begin{Bmatrix}\Psi_1^{(2)}\\ A_1^{(2)}\end{Bmatrix}\right| = |A_e^{(2)}(u,v)|\left[\frac{\sinh^2\rho_0^{(2)}+\sin^2\Phi_0^{(2)}}{\sinh^2(\alpha^{(2)}s+\rho_0^{(2)}+\rho_s^{(2)})+\sin^2(\beta_s^{(2)}+\Phi_0^{(2)}+\Phi_s^{(2)})}\right]^{\frac{1}{2}} \cdot\left\{\begin{matrix}1/|Z_{cm}|\,[\sinh^2(\alpha^{(2)}w+\rho_s^{(2)})+\sin^2(\beta^{(2)}w+\Phi_s^{(2)})]^{\frac{1}{2}}\\ [\sinh^2(\alpha^{(2)}w+\rho_s^{(2)})+\cos^2(\beta^{(2)}w+\Phi_s^{(2)})]^{\frac{1}{2}}\end{matrix}\right\}, \quad (60)$$

and

$$\left|\begin{Bmatrix}A_1^{(1)}\\ \Psi_1^{(1)}\end{Bmatrix}\right| = |\Psi_e^{(1)}(u,v)|\left[\frac{\sinh^2\rho_0^{(1)}+\sin^2\Phi_0^{(1)}}{\sinh^2(\alpha^{(1)}s+\rho_0^{(1)}+\rho_s^{(1)})+\sin^2(\beta^{(1)}s+\Phi_0^{(1)}+\Phi_s^{(1)})}\right]^{\frac{1}{2}} \cdot\left\{\begin{matrix}1/|Z_{cm}|\,[\sinh^2(\alpha^{(1)}w+\rho_s^{(1)})+\cos^2(\beta^{(1)}w+\Phi_s^{(1)})]^{\frac{1}{2}}\\ [\sinh^2(\alpha^{(1)}w+\rho_s^{(1)})+\sin^2(\beta^{(1)}w+\Phi_s^{(1)}]^{\frac{1}{2}}\end{matrix}\right\}. \quad (61)$$

The condition for resonance (maximum values of the amplitudes) is in all cases

$$\beta^{(q)}s_n+\Phi_0^{(q)}+\Phi_s^{(q)}=n\pi, \quad (62)$$

provided

$$[(\alpha^{(q)}/\beta^{(q)})\sinh(\alpha^{(q)}s+\rho_0^{(q)}+\rho_s^{(q)})]^2\ll 1. \quad (63)$$

The width $\Delta s_n^{(q)}$ of a resonance curve between the lengths for which the amplitudes $|\Psi|$, $|A|$, etc., have $2^{-1/2}$ of the maximum value may be obtained from (60) and (61). It is

$$\Delta s_n^{(q)}=2(\alpha^{(q)}s_n+\rho_0^{(q)}+\rho_s^{(q)})/\beta^{(q)}, \quad (64)$$

provided

$$(\Delta s_n^{(q)}/2)^2\ll(s_n^{(q)})^2. \quad (65)$$

THE PROPAGATION CONSTANT

IV

The propagation constant $K_g^{(q)}$ has been given by Eq. (20), where $k^2=\omega^2\mu\epsilon^*$ [*vide* (2)]. Let us examine the shape-factor $(\gamma^{(q)})$, which by (57) is in general complex. As we have seen from (55) and (57) this complexity is the result of the finite conductivity of the metal walls of the pipe. Omitting the superscript (q) for brevity, since these results hold for either type of oscillation, we may write

$$\gamma^2=(\gamma_r+i\gamma_i)^2=\bar{\gamma}^2(1+ih_c), \quad (66)$$

where

$$\bar{\gamma}^2\equiv\gamma_r^2-\gamma_i^2; \quad h_c\equiv 2\gamma_r\gamma_i/\bar{\gamma}^2. \quad (67)$$

Then we have

$$K_g^2=(\bar{\gamma}^2-\beta_d^2)+i(h_c\bar{\gamma}^2+h_d\beta_d^2), \quad (68)$$

and

$$\beta_d^2\equiv\omega^2\mu_d\epsilon_d; \quad h_d\equiv\sigma_d/\omega\epsilon_d, \quad (69)$$

where the subscript d denotes the dielectric and the subscript c refers to the conductor. If we make the further definition

$$\nu^2\equiv\bar{\gamma}^2/\beta_d^2, \quad (70)$$

then (68) becomes

$$K_g^2=-\beta_d^2(1-\nu^2)\left(1-i\left[\frac{h_d+h_c\nu^2}{1-\nu^2}\right]\right). \quad (71)$$

We have also the following relations between the phase and group velocities v_ϕ and v_G, and

[14] L. J. Chu and W. L. Barrow, Proc. I. R. E. **26** (December, 1938).

[15] G. C. Southworth, Proc. I. R. E. **25**, 807 (1937).

[16] Reference 10, Chapter VIII.

[17] J. G. Brainerd, G. Koehler, H. J. Reich, L. F. Woodruff, *Ultra-High Frequency Techniques* (D. Van Nostrand Company, Inc., New York, 1942), seventh printing. This book contains an extensive bibliography in Chapter 16. See also a *Reference Guide to Ultra High Frequencies*, E. Kelsey (Zenith Radio Corporation, Chicago, 1942); E. U. Condon, Rev. Mod. Phys. **14**, 341 (1942).

the wave-length in the guide λ_g:

$$v_\phi=\omega/\beta_g;\quad v_G=\partial\omega/\partial\beta_g;$$
$$\beta_g=2\pi/\lambda_g;\quad v_\phi v_G=c_d^2. \tag{72}$$

Here c_d $(=\mu_d\epsilon_d)^{-\frac{1}{2}}$ is the velocity of propagation of plane waves in medium d and β_g is the phase shift, obtained from (20) and (71) and given explicitly below. We have three cases to distinguish.

Case I. Transmission, $\nu^2<1$: From (71) we have

$$K_g=\alpha_g+i\beta_g=\beta_d(1-\nu^2)^{\frac{1}{2}}[g(h)+if(h)],$$
$$h\equiv(h_d+h_c\nu^2)/(1-\nu^2), \tag{73}$$

where $f(h)$ and $g(h)$ are functions tabulated originally by G. W. Pierce,[18] and are defined as

$$f(h)\equiv\{[(1+h^2)^{\frac{1}{2}}+1]/2\}^{\frac{1}{2}}$$
$$=1+h^2/8-5h^4/128+\cdots, \tag{74}$$
$$g(h)\equiv\{[(1+h^2)^{\frac{1}{2}}-1]/2\}^{\frac{1}{2}}$$
$$=(h/2)(1-h^2/8+9h^4/128-\cdots). \tag{75}$$

From Eq. (73) we obtain the general expression for the attenuation and phase constants

$$\alpha_g=\beta_d(1-\nu^2)^{\frac{1}{2}}g(h);\quad \beta_g=\beta_d(1-\nu^2)^{\frac{1}{2}}f(h). \tag{76}$$

For perfect conductors and dielectrics $h_c=h_d=0$, and consequently $g(h)=0$; $f(h)=1$. With these values it is a simple matter to determine α_g, β_g, ν, v_ϕ, v_G from (76), (72), (70), (67). By far the most important case is that for low dielectric losses and good conductors. This requires that

$$h^2/8\ll1, \tag{77}$$

as can be seen at once from (74). Then we have

$$\alpha_g=\alpha_{gc}+\alpha_{gd}=\frac{\beta_d\nu^2h_c}{2(1-\nu^2)^{\frac{1}{2}}}+\frac{\beta_dh_d}{2(1-\nu^2)^{\frac{1}{2}}}, \tag{78}$$

and it follows immediately that $(\alpha_g/\beta_g)^2\ll1$, which is the usual condition for good transmission circuits. It is quite valid to consider the attenuation as a linear combination (78) of the separate attenuations due to the dielectric and tube walls, provided the condition (77) is fulfilled. This linearity is commonly assumed in practice but does not seem to have been clearly demonstrated previously. For poor dielectrics and metal walls higher powers of h, cf. (74), are no longer negligible, and the linear relation (78) is not then correct. In this case we must use (74) and (75), with the aid of Pierce's tables.[18]

For good dielectrics and conductors we have

$$\alpha_{gd}=(\mu_d/\epsilon_d)^{\frac{1}{2}}\sigma_d/2(1-\nu^2)^{\frac{1}{2}};$$
$$\beta_g=\beta_d(1-\nu^2)^{\frac{1}{2}};\quad \nu=\gamma_r^2/\beta_d^2; \tag{79}$$
$$v_\phi=c_d(1-v^2)^{-\frac{1}{2}};\quad v_G=c_d(1-v^2)^{\frac{1}{2}}. \tag{80}$$

Case II. Cut-Off, $\nu=1$: In this instance (68) reduces, with the aid of (70), to

$$K_g=\beta_d(h_d+h_c)^{\frac{1}{2}}(1+i)/2, \tag{81}$$

from which it follows at once that

$$\alpha_g=\beta_g=\beta_d(h_d+h_c)^{\frac{1}{2}}/2, \tag{82}$$
$$v_\phi=c_d/v_G=c_d[(h_c+h_d)/2]^{-\frac{1}{2}}. \tag{83}$$

For low attenuation h_c, $h_d\ll1$, consequently the phase velocity v_ϕ exceeds the free space velocity c, which in turn is greater than the velocity c_d in the dielectric d. For perfect media the phase velocity becomes infinite, and the attenuation and phase shift both vanish.

Case III. Attenuation, $\nu^2>1$: Equation (71) now becomes

$$K_g^2=\beta_d^2(\nu^2-1)(1+ih'), \tag{84}$$

with

$$h'\equiv(h_c\nu^2+h_d)/(\nu^2-1).$$

Similarly it follows at once that

$$\alpha_g=\beta_d(\nu^2-1)^{\frac{1}{2}}f(h');\quad \beta_g=\beta_d(\nu^2-1)^{\frac{1}{2}}g(h'), \tag{85}$$

and for good conductors and dielectrics $h'^2/8\ll1$, so that

$$\alpha_g=\beta_d(\nu^2-1)^{\frac{1}{2}};\quad \beta_g=\beta_d(\nu^2-1)^{\frac{1}{2}}h'/2. \tag{86}$$

Here the attenuation is large and the phase shift small; in fact, for perfect conductors the wavelength λ_g in the pipe is infinite, and there is no wave phenomenon at all. The cut-off is sharp, unlike the gradual cut-off for the case of imperfect media.

For good conductors α_{gc} may be determined[9,10] by integrating the normal component of the complex Poynting vector[19] $\mathbf{S}=(\mathbf{E}\times\mathbf{H}^*)/2$ over a *closed*

[18] G. W. Pierce, Proc. Am. Acad. Arts and Sci. **57**, 175–191 (April, 1922).

[19] Reference 7, pp. 135–137.

surface whose ends are planes perpendicular to the axis of the pipe and whose sides are the inner pipe walls. The approximation made requires that the surface current of the good conductor be considered essentially the same as that for a perfect one, and is in accord with the assumption that $(h^2/8)\ll 1$. Many different (but equivalent) expressions for α_{gc} have been obtained,[9,10] and so we will content ourselves with listing only two forms here, to which we shall have some recourse in Part II for the purposes of calculation. A detailed discussion of the variation of α_g with frequency and the cross-sectional dimensions of the tube is beyond the scope of this paper.

For the H modes of a rectangular pipe of width a and height b we have

$$\alpha_{gc}=\frac{2r_s}{b}\left(\frac{\epsilon_c}{\mu_c}\right)^{\frac{1}{2}}(1-\nu^2)^{-\frac{1}{2}} \times\left[\frac{(1-\nu^2)p(pn_x^2+n_y^2)}{p^2n_x^2+n_y^2}+\nu^2(1+p)\right], \quad (87)$$

where $p=a/b$ and n_x and n_y are the number of half-wave-lengths in the x and y directions. The quantities ϵ_c, μ_c, and σ_c are the absolute permittivity, permeability, and conductivity of the pipe walls, and r_s is the surface impedance obtained from the usual skin thickness formulae:[20]

$$r_s=(\omega\mu_c/2\sigma_c)^{\frac{1}{2}}. \quad (88)$$

For the H modes in a circular pipe of radius a we have

$$\alpha_{gc}=\frac{r_s}{a}\left(\frac{\epsilon_c}{\mu_c}\right)^{\frac{1}{2}}(1-\nu^2)^{-\frac{1}{2}}\left[\frac{n^2}{(a\gamma_{r,(n,p)})^2-n^2}+\nu^2\right], \quad (89)$$

where, following the notation of (52), n is the order of the Bessel function and p is the number of the root of the equation from which $(\gamma_{np})_{\text{real}}$ is obtained. The attenuation α_{gd} is found at once from (68) and (78).

THE CHARACTERISTIC IMPEDANCES

V

Unlike the conventional line, there are two possible definitions of characteristic impedance for the wave guide, one for the E and H modes, respectively. Both expressions are obtained by formal analogy with the ordinary theory. Before discussing the character of this analogy let us examine the analytical behavior of these impedances. From (23) and (24) we have

$$Z_{ce}=K_g^{(1)}/Y_e\equiv R_{ce}(1-i\phi_{ce}), \quad (90)$$

$$Z_{cm}=Z_m/K_g^{(2)}\equiv R_{cm}(1-i\phi_{cm}). \quad (91)$$

Dropping the subscript **g** we have from the previous section

$$(q=1)\qquad R_{ce}=\frac{\beta^{(1)}+\alpha^{(1)}h_d^{(1)}}{\omega\epsilon(1+h_d^{(1)2})}; \qquad \phi_{ce}=\frac{\alpha^{(1)}-\beta^{(1)}h_d^{(1)}}{\beta^{(1)}+\alpha^{(1)}h_d^{(1)}}, \quad (92)$$

and

$$(q=2)\quad R_{cm}=\frac{\omega\mu\beta^{(2)}}{\alpha^{(2)2}+\beta^{(2)2}};\quad \phi_{cm}=\alpha^{(2)}/\beta^{(2)}, \quad (93)$$

and

$$X_c\{^e_m\}=-\phi_c\{^e_m\}R_c\{^e_m\}, \quad (94)$$

where the upper line applies as $q=1$ and the lower for $q=2$.

The H modes yield simpler expressions for the characteristic impedance, since the dependence on the imperfect dielectric h_d does not appear explicitly in R_{cm} or ϕ_{cm} and occurs only in α_g. This follows from the fact that for the H modes there is no primary axial **E** field (although a small one does exist in the instance of imperfect conductors and dielectrics), and consequently there is no appreciable displacement of free charge in the direction of the tube's axis, a simpler state of affairs than for the E modes, which have both longitudinal and transverse dielectric currents. From the dependency of $Z_c\{^e_m\}$ on $K_g^{(q)}$ we must then distinguish three cases.

Case I. Transmission, $\nu^2<1$: The general relation for Z_{ce} and Z_{cm} may be obtained from (73) and (90), (91). We shall consider here only the case of good conductors and dielectrics, i.e., $\alpha^2\ll\beta^2$, $h_d^2\ll 1$. We have

$$R_{ce}=\left(\frac{\mu}{\epsilon}\right)^{\frac{1}{2}}(1-\nu^2)^{\frac{1}{2}};\quad \phi_{ce}=\alpha^{(1)}/\beta^{(1)}-h_d^{(1)},\quad \phi_{ce}^2\ll R_{ce}^2, \quad (95)$$

and

$$R_{cm}=\left(\frac{\mu}{\epsilon}\right)^{\frac{1}{2}}(1-\nu^2)^{-\frac{1}{2}};\quad \phi_{cm}=\alpha^{(2)}/\beta^{(2)},\quad \phi_{cm}^2\ll R_{cm}^2. \quad (96)$$

[20] Reference 7, Chapter IX, Sec. 9: 17 (46).

For perfect conductors and dielectrics ϕ_{ce} and ϕ_{cm} vanish.

Case II. Cut-Off, $\nu^2=1$: From (81) we have

$$R_{ce}=\left[\frac{\mu}{\epsilon}(h_c^{(1)}+h_d^{(1)})/2\right]^{\frac{1}{2}}(1+h_d^{(1)})/(1+h_d^{(1)2})$$

$$\doteq\left[\frac{\mu}{\epsilon}(h_c^{(1)}+h_d^{(1)})/2\right]^{\frac{1}{2}}, \quad (97)$$

$$\phi_{ce}=(1-h_d^{(1)})/(1+h_d^{(1)}), \quad (98)$$

$$R_{cm}=\left[\frac{\mu}{\epsilon}(h_d^{(2)}+h_c^{(2)})/2\right]^{-\frac{1}{2}}; \quad \phi_{cm}=1. \quad (99)$$

For perfect conductors and dielectrics R_{cm} becomes infinite, R_{ce} vanishes, and ϕ_{ce} equals $\phi_{cm}=1$.

Case III. Attenuation, $\nu^2>1$: Here we have from (84)–(86)

$$R_{ce}=\left[\frac{\mu}{\epsilon}(\nu^2-1)\right]^{\frac{1}{2}}\left(\frac{h^{(1)\prime}}{2}+h_d^{(1)}\right);$$

$$\phi_{ce}=\left[\frac{h^{(1)\prime}}{2}+h_d^{(2)}\right]^{-1}, \quad (100)$$

and

$$R_{cm}=\left(\frac{\mu}{\epsilon}\right)^{\frac{1}{2}}\frac{h^{(2)\prime}}{2}(\nu^2-1)^{-\frac{1}{2}};\phi_{cm}=(h^{(2)\prime}/2)^{-1}, \quad (101)$$

and for perfect materials $R_{ce}=R_{cm}=0$, and $\phi_{cm}=\phi_{ce}=\infty$. The reactances $X_c\{^e_m\}$ may be obtained from (92)–(101) with the aid of (94). The above results are summarized briefly in Table I.[21]

TABLE I.

ν	Z_{ce}	Z_{cm}
Trans: $\nu^2<1$	$\doteq R_{ce}$	$\doteq R_{cm}$
Cut-off: $\nu^2=1$	$\doteq 0$	$\doteq 0$
Atten: $\nu^2>1$	$\doteq iX_{ce}<0$	$\doteq iX_{cm}>0$

CONCLUSIONS

VI

As we have seen in the previous work, there is a formal analogy between potential and stream functions and the impedances of the wave guide theory with the potential difference V, total current I, and impedances as given in ordinary transmission line work. The wave guide problem is three-dimensional, whereas the conventional line is essentially one-dimensional. Since it is always possible to separate the axial variable z from the transverse variables u, v, the correspondence between the conventional theory and that discussed here for hollow pipes of various cross sections is seen to lie in the fact that the dependence on the axial variable z is the same in both cases, and thus the properties of one go over without change into the properties of the other provided they are functions of z only. The propagation constant $K^{(q)}$, however, depends in part on the transverse nature of the transmission circuit and on the mode excited, so that in general, *unless we specify a particular mode and a particular cross section of pipe*, the analogy between the two theories is only formal.

This formal equivalence is not physically complete, for as we see from (18), (36), (37), (51), and (52), the stream functions do not reduce to the total current I in the limit of very small cross sections since $T(u, v)$ does not then reduce to unity. The total currents are still defined in the dielectric between the conducting walls, and the surface currents must be obtained from them by differentiation. It may easily be shown from (36) or (37) that when the circuit is infinitely long, the quantities $Z_c\{^e_m\}$ are correctly defined (just as V/I for an infinite line yields Z_c for the line), *for a given mode and a given pipe.* In this case we may reasonably speak of an attenuation constant α_g, phase shift β_g, wavelength in the guide λ_g, and $\Phi_{0,s}^{(q)}$ and $\rho_{,s}^{(q)}$. Their physical interpretation is the same as for the corresponding quantities of the conventional line. Thus the familiar l, c, g, r (loop inductance, capacitance, conductance, and resistance) of the line may have their counterparts in the instance of the hollow pipe, but subject to the cross-sectional dependence mentioned above. When the termination is a device such that $\rho_s=\infty$, the circuit, as before, will be characterized by traveling waves and a standing wave ratio of unity. The characteristic impedances $Z_c\{^e_m\}$ for all coaxial or hollow tubes reduce to an expression which is independent of the transverse dimensions of the pipe, and is $(\mu/\epsilon)^{\frac{1}{2}}$, which has the value 376.7 ohms if the dielectric is air. This

[21] S. A. Schelkunoff, Proc. I. R. E. **25**, 1457 (1937).

means that for matching purposes it is *not* sufficient to equate $Z_c\{^{e}_{m}\}$ of the two guides (as it was for two lines), since for tubes of different cross sections and excited in different modes the driven tube will quite obviously not have a unit standing wave ratio. Matching may be achieved, as before, when the standing wave ratio is one, but to accomplish this will require adjustments of the line variables which are not contained in $Z_c\{^{e}_{m}\}$ alone.

In a similar fashion the terminal impedances $Z_0^{(q)}$, $Z_s^{(q)}$ are physically analogous to the terminal impedances defined for the conventional theory,[1,11] but only for a particular mode and pipe. The definitions of impedance (28), (30), (33), and (35) are not, however, generalizations of Z_0, Z_s of the ordinary two-wire or coaxial line, as explained above, since $Z_0^{(q)}$ and $Z_s^{(q)}$ do not reduce in the limiting case to Z_0 or Z_s, but are dependent on the distribution of current within the pipe, i.e., on the stream functions, and not on the total current, as are Z_0 and Z_s. The total current in these cases is of little significance in characterizing the behavior of the guide.

We have already developed in some detail the formal reduction of the wave guide problem to that of the conventional line. From the standpoint of practice the most important results are that—

(1) the formulae of the ordinary theory are applicable to wave guides and cavity resonators, subject to the restrictions and interpretations discussed above;

(2) since it is possible to determine $\rho_s^{(q)}$ and $\Phi_s^{(q)}$ experimentally from *any* termination (as will be mentioned briefly in Part II), the general relations of the usual theory may then be used to obtain specific experimental and theoretical answers to an almost unrestricted number of transmission circuit problems involving wave guides and resonators. Our success in obtaining analytical expressions for various terminations will depend on our ability to determine the field distributions, which is an extremely difficult problem for even the simplest geometries, and can be successfully treated in but a few cases. However, meaningful results may always be obtained experimentally in terms of $\rho_s^{(q)}$ and $\Phi_s^{(q)}$.

In Part II we shall consider a few practical applications of the theory, with several experimental examples and an analytical evaluation of the terminal and input impedances of a closed guide or resonator containing several sections of imperfect dielectric material at one end.

Transmission Line Theory Applied to Wave Guides and Cavity Resonators. II

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In Part II a number of practical applications of the theory outlined in Part I are considered. Formulae for power transfer, efficiency of transmission, standing wave ratios, and insulating disks are given, and the experimental determination of the line and terminal parameters is treated briefly. A number of examples illustrating the theory are discussed: (1) a rectangular wave guide terminated by a section of pipe containing a horizontal slit at one end and a flat horn of 45° aperture at the other, (2) the same guide terminated by a section of tube with a horizontal slit at one end and opening into space at the other, (3) the rectangular guide with a variable horizontal and vertical slit, in either case followed by a closed quarter-wave section of pipe, (4) a closed cylindrical tube containing several transverse slabs of imperfect dielectric material. In (1)–(3) the terminal functions ρ and Φ are determined experimentally, and in (4) analytical expressions for the terminal and input impedance are obtained. Experimental and theoretical curves illustrating the examples are included.

VII

SPACE does not permit a detailed discussion of the many applications of the formulae of Part I to the practical problems involving transmission circuits. The actual derivations used in the applications outlined below follow immedi-

ately from the theory of the previous sections.* The development given in the above reference pertains primarily only to parallel and coaxial lines of transverse dimensions small compared with a wave-length, but, as indicated in Part I, these results may also be applied, after some modification, to circuits consisting of wave guides and cavity resonators. Some of the formulae are listed here for convenience.

(a) The Transfer of Power and Efficiency of Transmission

A wave guide of length s has a load impedance $Z_s^{(q)}$ given by (40). The power supplied at the input end of the circuit is

$$P_0^{(q)} = \left\{ \begin{matrix} |A_0^{(1)}|^2 \\ |\Psi_0^{(2)}|^2 \end{matrix} \right\} R_{in}^{(q)}, \quad q=1, 2, \quad (102)$$

where $A_0^{(1)}$ and $\Psi_0^{(2)}$ are given by (37) and (36) suitably specialized at $z=0$. The power supplied to the load is

$$P_s^{(q)} = \left\{ \begin{matrix} |A_s^{(1)}|^2 \\ |\Psi_s^{(2)}|^2 \end{matrix} \right\} R_s^{(q)}, \quad q=1, 2, \quad (103)$$

with the help again of (36) and (37), at $z=s$. We have finally

$$\left(\frac{P_s^{(q)}}{P_0^{(q)}}\right) = \frac{\sinh 2\rho_s^{(q)} - \phi_c\{_m^e\} \sin 2\Phi_s^{(q)}}{\sinh 2(\alpha^{(q)}s+\rho_s^{(q)}) - \phi_c\{_m^e\} \sin 2(\beta^{(q)}s+\Phi_s^{(q)})}. \quad (104)$$

From this it is a simple matter to compute $(P_s^{(q)}/P_0^{(q)})$. In all practical cases of large loads $\rho_s^{(q)}$ is sufficiently large so that we may neglect terms in $\phi_c\{_m^e\}$, and (104) becomes

$$(P_s^{(q)}/P_0^{(q)}) = \frac{\sinh 2\rho_s^{(q)}}{\sinh 2(\alpha^{(q)}s+\rho_s^{(q)})}, \quad (105)$$

provided

$$\sinh 2\rho_s^{(q)} \gg \phi_c\{_m^e\}. \quad (106)$$

For maximum efficiency we find

$$\rho_s^{(q)} = \infty,$$

and

$$\therefore \quad P_s^{(q)}/P_0^{(q)} = \exp(-2\alpha^{(q)}s). \quad (107)$$

* The reader is also referred to R. King, J. App. Phys. **14**, 593–599 (1943), for a more complete consideration of such topics as power transfer, efficiency of transmission, standing wave ratios, impedance matching, insulating stubs, and the experimental determination of the line and terminal parameters.

(b) Standing Wave Ratios

The standing wave ratios $S_{\text{stream}}^{(q)}$, $S_{\text{potential}}^{(q)}$, may be defined as follows, by analogy with the conventional line theory, in terms of the stream and potential function ratios:

$$\begin{aligned} S_s^{(1)} &= |A_1^{(1)}|_{\max}/|A_1^{(1)}|_{\min}; \\ S_s^{(2)} &= |\Psi_1^{(2)}|_{\max}/|\Psi_1^{(2)}|_{\min}, \end{aligned} \quad (108)$$

and

$$\begin{aligned} S_p^{(1)} &= |\Psi_1^{(1)}|_{\max}/|\Psi_1^{(1)}|_{\min}; \\ S_p^{(2)} &= |A_1^{(2)}|_{\max}/|A_1^{(2)}|_{\min}. \end{aligned} \quad (109)$$

These ratios are equivalent to a comparison of $|E|_{\max}/|E|_{\min}$, $|H|_{\max}/|H|_{\min}$ or any quantities whose maximum and minimum values may be measured by a properly inserted loop or straight-wire probe, which is sufficiently loosely coupled to the circuit so that it does not disturb the distribution of the quantities measured. Unless otherwise specified, it will be understood that the distance $w=s-z$ is to be kept as small as possible, to keep $S_{s,p}^{(q)}$ single valued. Using (60) and (61) maximized and minimized with respect to w we obtain

$$S_s^{(q)} = \frac{\cosh(\alpha^{(q)}w_{\max}^{(q)}+\rho_s^{(q)})}{\sinh(\alpha^{(q)}w_{\min}^{(q)}+\rho_s^{(q)})} = S_p^{(q)}, \quad (110)$$

where for $S_p^{(1)}$ and $S_s^{(2)}$ with n integral

$$w_{\min}^{(q)} = n\lambda_g/2 - \Phi_s^{(q)}/\beta^{(q)}, \quad (111)$$

and

$$w_{\max}^{(q)} = \left(\frac{2n+1}{4}\right)\lambda_g - \Phi_s^{(q)}/\beta^{(q)}, \quad (112)$$

with n chosen (independently) as small as possible in each case. For $S_p^{(2)}$ we must interchange the subscripts max. and min. in (111), (112), and similarly for $S_s^{(1)}$. For low loss tubes, i.e., pipes with low over-all attenuation which are used to drive a power-absorbing load, $w_{\min}^{(q)}$ and $w_{\max}^{(q)}$ are always small enough so that we can write

$$(\alpha^{(q)}w_{\min,\max}^{(q)}) \ll \rho_s^{(q)}, \quad (113)$$

and

$$\therefore \quad S_{s,p}^{(q)} = \coth \rho_s^{(q)}, \quad (114)$$

which determines $\rho_s^{(q)}$ if $S_{s,p}^{(q)}$ is known.

(c) Impedance Matching and Insulating Disks

The problem of matching two sections of a guide, or inserting a section of tube between a

load impedance $Z_s^{(q)}$ and a long pipe cannot be solved in general by any of the analysis of Part I, except in the elementary case where the two guides have identical cross sections and are excited in the same mode. As was pointed out in Section VI, the characteristic impedances $Z_c\{^e_m\}$ are not capable of the analogous physical interpretation of conventional transmission line theory when the pipes are dissimilar, and consequently one cannot equate, for example, $Z_c\{^e_m\}$ of one section to the input impedance of a given length of pipe with a load $Z_s^{(q)}$ and obtain satisfactory results. Matching may be achieved experimentally, however. Consider again, for example, the problem of matching a rectangular guide supplying power to a circular guide excited in the same mode and terminated in an arbitrary fashion. One simply determines $A_s^{(q)}$ and $F_s^{(q)}$ for the terminated pipe; then we may treat this entire pipe as the termination of the other, whereupon $A_s^{(q)}$ and $F_s^{(q)}$ for the circular guide become $\rho_s^{(q)}$ and $\Phi_s^{(q)}$ for the rectangular guide. With suitable slits in the rectangular tube to vary the reactance and a sliding section of pipe to vary the resistance (as well as the reactance), it is possible to obtain a match, which as before is characterized by a standing wave ratio of unity.

At high frequencies it is often very convenient to be able to use high impedance sections of tubes. Such sections are easily obtained, through the use of a metal piston at the end of the guide. If the over-all attenuation [or $A_s^{(q)2}$ and $\phi_c\{^e_m\}^2$] is small compared to unity, as is generally the case in practical circuits, the maximum input impedance of a section of length s is

$$(R_{in}^{(q)})_{\max} = R_c\{^e_m\}/[\alpha^{(q)}s + \rho_s^{(q)}], \quad (115)$$

which occurs at

$$\beta^{(q)}s = n\pi - \Phi_s^{(q)}. \quad (116)$$

Thus the highest resistance for a given guide occurs when the over-all attenuation of the section is smallest, i.e., when the denominator of (115) is reduced as much as possible. The conditions for a piston are then from (43) and (44)

$$\rho_s^{(q)} \doteq 0; \quad \Phi_s^{(q)} = \pi/2, \quad (117)$$

where, for $n=1$, we have

$$(R_{in}^{(q)})_{\max} = R_c\{^e_m\}/\alpha^{(q)}s', \quad (118)$$

where

$$s' = \pi/2\beta^{(q)} = \lambda_g^{(q)}/4, \quad (119)$$

with $\lambda_g^{(q)}$ the wave-length in the pipe.

(d) Experimental Determination of the Line and Termination Parameters

For a more detailed discussion of this topic see Part I, reference 1.* We list here only the relevant formulae, with a brief description:

$$\lambda_g = 2(s_2 - s_1), \quad (120)$$

where s_1 and s_2 represent the location of two successive field maxima [or maxima of $A_1^{(q)}$ or $\Psi_1^{(q)}$], provided the width of the resonance peaks are small so that (65) is true. For the measurement of the attenuation constant of the circuit we have

$$\alpha^{(q)} = \beta^{(q)}(\Delta s_n - \Delta s_1)/2(s_n - s_1), \quad (121)$$

where Δs_1 and Δs_n are the widths of two resonance peaks at 0.707 of maximum and $s_n - s_1$ is the distance between them. The position of the detector and source must not be changed during successive measurements, nor must the degree of coupling be altered.

The phase functions $\Phi_{s,0}^{(q)}$ of a termination are determined readily from (62) by

$$\Phi_{s,0}^{(q)} = \pi/2 + \beta^{(q)}(s_1 - s_{1\Phi}), \quad (122)$$

where s_1 is the resonant length of the guide terminated by a piston and $s_{1\Phi}$ the resonant length when the unknown termination is added. Condition (63) must be satisfied in this case.

The attenuation function is obtained from

$$\rho_s^{(q)} = \beta^{(q)}(\Delta s_{1\rho} - \Delta s_1)/2 + \alpha^{(q)}(s_1 - s_{1\rho}), \quad (123)$$

where Δs_1 is the width of the first resonance curve with the metal piston terminating the guide ($\rho_{s,0}^{(q)} \doteq 0$), and $\Delta s_{1\rho}$ is the width of the first resonance curve with the termination of unknown $\rho_s^{(q)}$; s_1 and $s_{1\rho}$ are the respective resonant lengths of the two cases. As before it is very important that the degree of coupling be unchanged during the measurement, as well as the location of the source and detector with respect to the end at $z=0$.

* Part I, reference 1, pp. 598–599.

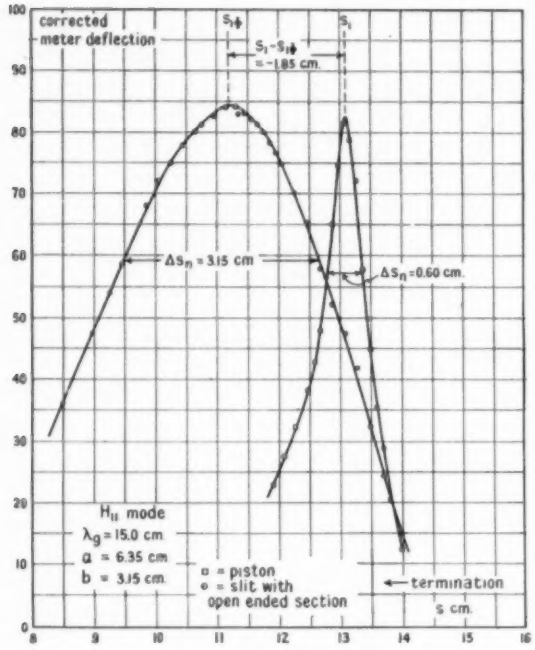


FIG. 1. Resonance curves obtained for a rectangular brass pipe excited in the H_{11} mode and terminated in a horizontal slit 1.2 cm wide, with 11.5 cm of pipe extending beyond to an open end. The termination is capacitative and is characterized by $\rho_s^{(2)}=0.54$ and $\Phi_s^{(2)}=0.79$ rad.

The Q of a resonant guide or cavity may be defined in the customary way by

$$Q^{(q)}=f_n^{(q)}/2\delta f^{(q)}=s_n^{(q)}/\Delta s_n^{(q)}, \qquad (124)$$

provided (65) is satisfied or $(\delta f)^2 \ll f_n^2$. Here δf is the change in frequency required to reduce the resonant amplitude of the fields to $2^{-\frac{1}{2}}$ of their maximum at f_n. From (64) we have

$$Q^{(q)}=\beta^{(q)}s_n^{(q)}/2(\alpha^{(q)}s_n^{(q)}+\rho_0^{(q)}+\rho_s^{(q)}). \qquad (125)$$

For $\rho_{0,s}^{(q)} \doteq 0$, we have

$$Q^{(q)}=\beta^{(q)}/2\alpha^{(q)}. \qquad (126)$$

(e) Examples

Let us consider a number of practical examples which are not easily handled by analytical methods but which are immediately capable of experimental treatment.

To illustrate the case when $\rho_s^{(q)}$ is small we have a long rectangular guide excited in the H_{11} mode and containing a source and detector at one end and a horizontal slot 1.2 cm wide at the other, followed by 11.5 cm of pipe leading to an open end (see Fig. 1). The guide contains a calibrated moveable piston at the end $z=0$. The source and detector are placed so as always to remain at the same distance from this end and from each other. This is important, as otherwise the attenuation and phase shift due to the source and the detector, which are contained in the terminal functions $\rho_0^{(q)}$ and $\Phi_0^{(q)}$, would vary with successive measurements and would not subtract out, as they do in (123). We would then be unable to determine $\rho_s^{(q)}$ and $\Phi_s^{(q)}$ accurately.

With pistons at both ends the attenuation of the circuit was measured with the help of (121) and found to be negligible, as was $\rho_s^{(2)}$ for the piston. From (120) and (123) it was found for the rectangular guide of width $a=6.35$ cm and height $b=3.15$ cm that

$$\begin{aligned} &\lambda_g=0.150 \text{ m}; \quad \Delta s_{n1}=0.006 \text{ m}; \\ &\beta_g=2\pi/\lambda_g=41.9 \text{ rad/m}; \\ &\rho_0^{(2)}=\beta_g\Delta s_{n1}/2=41.9(0.006)/2=0.125. \end{aligned} \qquad (127)$$

The termination to be tested was added and another resonance curve obtained from which it was found that

$$\Delta s_{n2}=0.0315 \text{ m and } s_{n1}-s_{n2}=-0.0185 \text{ m}, \qquad (128)$$

and from (122) and (123) we have

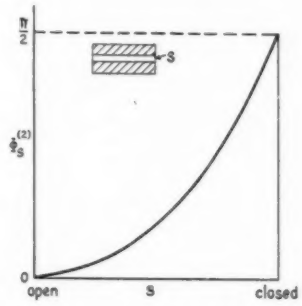
$$\begin{aligned} &\rho_s^{(2)}=41.9(0.0185)/2=0.54; \\ &\Phi_s^{(2)}=\pi/2-41.9(0.0185)/2=0.79 \text{ radian}. \end{aligned} \qquad (129)$$

Since $\Phi_s^{(2)}<\pi/2$ the termination is capacitative, as can be seen from Fig. 6, reference 1, or (42). Figure 1 is a plot of the data from which these terminal functions were obtained.

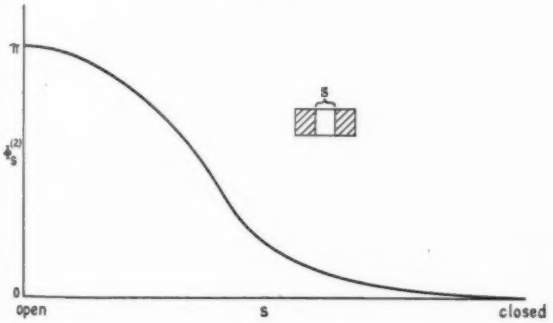
When $\rho_s^{(q)}$ is particularly large, 1 or greater, we cannot use (123), as (65) is not satisfied. For practical purposes the guide is then nearly equivalent to one terminated in its characteristic impedance. Values of $\Phi_s^{(q)}$ are not in this case required. To determine $\rho_s^{(q)}$ we measure the standing wave ratio and use (114). For the guide of the previous example, excited in the H_{11} mode and terminated by a horizontal slit followed by a telescoping section of line which opens into a flat horn of 45° aperture, the standing wave ratio was found to be 1.2, and from (114) we have

$$\rho_s^{(2)}=\coth^{-1}(1.2)=1.20. \qquad (130)$$

A third example consists of determining $\Phi_s^{(2)}$ for the same rectangular guide terminated first in an adjustable horizontal slit followed by a quarter-wave-length section of closed pipe, and then in a variable vertical slit, also followed by a closed quarter-wave section. The measurements were made in a fashion similar to the first example, with the help of (122), and in each case $\rho_s^{(q)}$ and $\alpha^{(q)}s$ were found to be negligible. The phase shift $\Phi_s^{(2)}$ is shown in Fig. 2 as a function of slit width.



(a) Horizontal slit



(b) Vertical slit

FIG. 2. The terminal functions $\Phi_s^{(2)}$ for a rectangular guide terminated in (*a*), a horizontal slit followed by a closed quarter-wave section of pipe, and (*b*), a vertical slit, followed by a closed quarter-wave section.

CYLINDRICAL WAVE GUIDES AND CAVITY RESONATORS CONTAINING SEVERAL IMPERFECT DIELECTRIC MEDIA

We will now determine the terminal and input impedance of a cylindrical wave guide containing several sections of imperfect dielectric materials at one end of the closed pipe, for the general case of forced oscillations from a closely-coupled source. The analysis is equally applicable to the case of loose coupling, where we may consider the region containing the source to be of negligible longitudinal extent (cf. Section III), i.e., a plane distribution, and then we may quite reasonably treat the pipe as a cavity resonator, when, of course, it is excited in one of its natural modes, either by an adjustment of the termination, the exciting frequency, or the length of the tube. In the cases of loose coupling the resonance condition is given by (62).

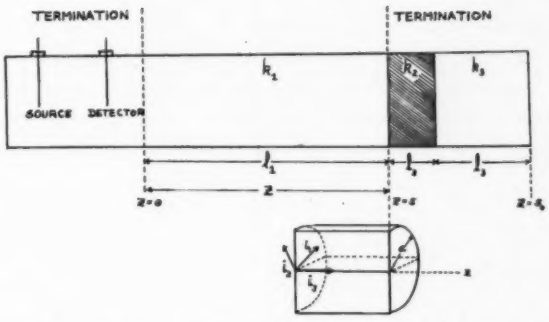


FIG. 3. Schematic representation of a wave guide or resonator containing a source and detector at one end and terminated at the other by several sections of imperfect dielectric material.

The circuit with its various dielectrics is shown schematically in Fig. 3, where the generator and detector are confined to a fairly small region at one end of the pipe, and the dielectrics are placed at the other. Another practical arrangement is to have the source and detector somewhere about the middle of the tube, and a slight modification of the analysis (see Section III) enables us to treat this case at once.

Although we are dealing with forced oscillations, the wave equations (11) and (12), specifying the fields in media one–three, are homogeneous, since the regions under consideration contain no sources (or sinks). The propagation constants for the various media ($j=1, 2, 3$) are given by Eq. (20). (See also Section IV.) We have remarked before (Sections III, IV) that the propagation constant $K^{(q)}$ depends on the geometry of the tube and on the type of mode excited, as well as on the electrical constants of the medium (k_d^2) and the frequency of the source. In the following analysis we will assume that the pipe walls are perfectly conducting, and only the dielectrics are imperfect. Since the contribution to the attenuation α_{gd} due to the dielectric may be added directly to that due to the metal walls, *viz.* (78), for imperfect but highly conducting metal, the rigourousness of our

theory is not materially affected in the practical case.* The development of Part I is quite general for all cross sections for which (18) is soluble. Here we will consider specifically only circular pipes.

The fields are expressible in terms of the vector stream and potential functions $\mathbf{A}^{(q)}$ by virtue of (5), (8), (10), (14)–(16). For the E modes ($q=1$) we have

$$\mathbf{E}^{(1)}=\left[\mathbf{i}_1\frac{\partial^2 A_z^{(1)}}{\partial r\partial z}+\mathbf{i}_2\frac{1}{r}\frac{\partial^2 A_z^{(1)}}{\partial\theta\partial z}+\mathbf{i}_3\gamma^{(1)2}A_z^{(1)}\right]\Big/ Y_e, \tag{131}$$

$$\mathbf{H}^{(1)}=\mathbf{i}_1\frac{\partial A_z^{(1)}}{r\partial\theta}-\mathbf{i}_2\frac{\partial A_z^{(1)}}{\partial r};\quad Y_e=i\omega\epsilon+\sigma. \tag{132}$$

Note that because of our assumption of perfect walls, $\gamma^{(q)}$ is real; see (57). For the H modes ($q=2$) the fields become

$$\mathbf{E}^{(2)}=-\mathbf{i}_1\frac{\partial A_z^{(2)}}{r\partial\theta}+\mathbf{i}_2\frac{\partial A_z^{(2)}}{\partial r}, \tag{133}$$

$$\mathbf{H}^{(2)}=\left[\mathbf{i}_1\frac{\partial^2 A_z^{(2)}}{\partial r\partial z}+\mathbf{i}_2\frac{\partial^2 A_z^{(2)}}{r\partial\theta\partial z}+\mathbf{i}_3\gamma^{(2)2}A_z^{(2)}\right]\Big/ Z_m, \tag{134}$$

$$Z_m=i\omega\mu,\quad (i=\sqrt{-1}).$$

The quantities $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ are unit vectors in the r, θ, z directions, respectively (see Fig. 3). We have then to satisfy the boundary conditions (53), (54) at the tube walls ($r=a$), and at the dielectric interfaces, $z=l_1, l_1+l_2, l_1+l_2+l_3$, where it can be shown with the help of the equation of continuity that the conditions (54) become

$$\mathbf{n}\cdot\epsilon_j{}^*\mathbf{E}_j^{(q)}=\mathbf{n}\cdot\epsilon_{j+1}^*\mathbf{E}_{j+1}^{(q)},\quad \epsilon^*=\epsilon-i\sigma/\omega,$$
$$\mathbf{n}\times\mathbf{H}_j^{(q)}=\mathbf{n}\times\mathbf{H}_{j+1}^{(q)};\quad j=1,2. \tag{135}$$

The surface charge and current densities on the cylindrical walls $r=a$ are given at once by (54). We find for both modes that

$$T^{(q)}(u,v)=J_n(\gamma_{np}^{(q)}r)\cos n\theta,\quad q=1,2, \tag{136}$$

is a sufficient solution for (18), where p indicates the pth root of the equations determining $\gamma_{np}^{(q)}$, i.e.,

$$J_n(\gamma_{np}^{(1)}a)=0;\quad J_n'(\gamma_{np}^{(2)}a)=0. \tag{137}$$

Typical solutions of (11) then take the form for medium one, $(j=1)$, $(0\leqq z\leqq l_1=s)$:

$$A_1{}^{(q)}(r,\theta,z)=\left(C_1{}^{(q)}\begin{Bmatrix}\cosh K_1{}^{(1)}z\\ \sinh K_1{}^{(2)}z\end{Bmatrix}+D_1{}^{(q)}\begin{Bmatrix}\sinh K_1{}^{(1)}z\\ \cosh K_1{}^{(2)}z\end{Bmatrix}\right)T^{(q)}(r,\theta), \tag{138}$$

for medium two, $j=2$, $(l_1\leqq z\leqq l_1+l_2)$:

$$A_2{}^{(q)}(r,\theta,z)=\left(C_2{}^{(q)}\begin{Bmatrix}\cosh K_2{}^{(1)}z\\ \sinh K_2{}^{(2)}z\end{Bmatrix}+D_2{}^{(q)}\begin{Bmatrix}\sinh K_2{}^{(1)}z\\ \cosh K_2{}^{(2)}z\end{Bmatrix}\right)T^{(q)}(r,\theta), \tag{139}$$

and for medium three, $j=3$, $(l_1+l_2\leqq z\leqq l_1+l_2+l_3=s_0)$:

$$A_3{}^{(q)}(r,\theta,z)=C_3{}^{(q)}\begin{Bmatrix}\cosh K_3{}^{(1)}(s_0-z)\\ \sinh K_3{}^{(2)}(s_0-z)\end{Bmatrix}T^{(q)}(r,\theta). \tag{140}$$

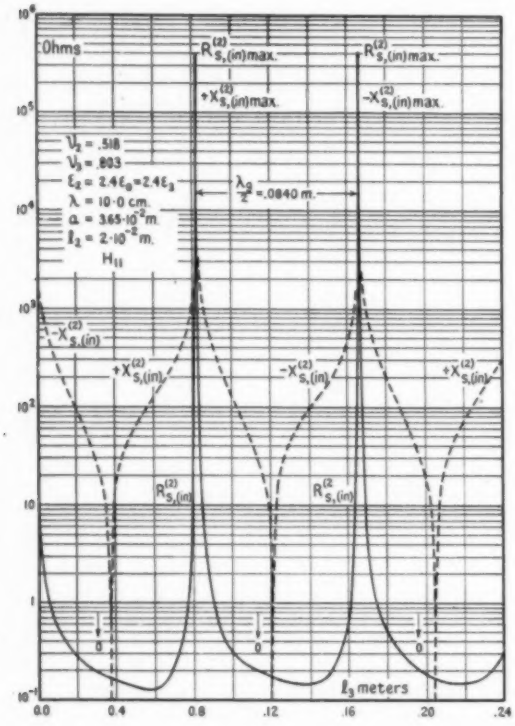


FIG. 4. Terminal impedance at $\lambda_g=16.8$ cm for a section of cylindrical brass pipe of radius $a=3.65$ cm, $\sigma_{\text{brass}}=1.43\ 10^7$ mhos/m, excited in the H_{11} mode and containing a thickness l_2 of dielectric ($\sigma_2=2\cdot10^{-4}$ mhos/m, $\epsilon=2.4\epsilon_0$), as a function of l_3. Here $\alpha_1^{(2)}=\alpha_3=2\cdot11\cdot10^{-3}$, $\alpha_2^{(2)}=2.93\cdot10^{-3}$ nepers/m, $\beta_2^{(2)}=83.4$ rad/m, $\beta_1^{(2)}=\beta_3^{(2)}=37.5$ rad/m. This also represents the input impedance of a guide or resonator containing a section of dielectric at one end ($l_3=0$) as a function of the tube length $s=l_3$.

* Note: To date there has been no completely rigorous treatment of the finite, imperfectly conducting cylinder.

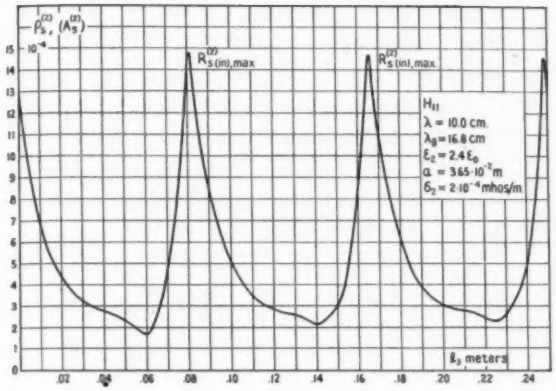


FIG. 5. The $\rho_s^{(2)}$ function for the termination of Fig. 4. This also represents $A_s^{(2)}$ for the hollow pipe of length l_3 terminated in a dielectric section l_2.

The complete solutions, of course, contain the time factor $e^{i\omega t}$ in product with (138)–(140). Here the upper and lower lines apply as $q=1, 2$, respectively. From the boundary conditions (53) and (135) at the various dielectric interfaces, and solving for $C_1^{(q)}/B_0^{(q)}$ and $D_1^{(q)}/B_0^{(q)}$ with the help of (30), (34), (138)–(140), we obtain finally for medium one, $q=1$:

$$A_1^{(1)}(r, \theta, z) = T^{(1)}(r, \theta) \times \left\{ \frac{\Gamma^{(1)} \sinh K_1^{(1)}(s-z) + \cosh K_1^{(1)}(s-z)}{\Gamma^{(1)} \sinh K_1^{(1)} s + \cosh K_1^{(1)} s} \right\}, \quad (141)$$

and for the H modes we have

$$A_1^{(2)}(r, \theta, z) = T^{(2)}(r, \theta) \times \left\{ \frac{\sinh K_1^{(2)}(s-z) + \Gamma^{(2)} \cosh K_1^{(2)}(s-z)}{\sinh K_1^{(2)} s + \Gamma^{(2)} \cosh K_1^{(2)} s} \right\}, \quad (142)$$

where

$$\Gamma^{(q)} = \left(\frac{\tanh K_2^{(q)} l_2 + \tau_2^{(q)} \tanh K_3^{(q)} l_3}{1 + \tau_2^{(q)} \tanh K_2^{(q)} l_2 \tanh K_3^{(q)} l_3} \right) \Big/ \tau_1^{(q)} \quad (143)$$

$$\equiv \bar{\Gamma}^{(q)} / \tau_1^{(q)}, \quad (143a)$$

and

$$\tau_1^{(1)} = \epsilon_2^* K_1^{(1)} / \epsilon_1^* K_2^{(1)}; \quad \tau_2^{(1)} = \epsilon_2^* K_3^{(1)} / \epsilon_3^* K_2^{(1)}, \quad (144)$$

$$\tau_1^{(2)} = \mu_1 K_2^{(2)} / \mu_2 K_1^{(2)}; \quad \tau_2^{(2)} = \mu_3 K_2^{(2)} / \mu_2 K_3^{(2)}. \quad (145)$$

By applying the appropriate analysis (26)–(35), we may obtain (36) and (37) and expressions for $Z_s^{(q)}$ in terms of $\Gamma^{(q)}$, where $T^{(q)}$ is given by (136) and $A_e^{(2)}$, $\Psi_e^{(1)}$ by (27) and (34). Media two and three form our termination, as can readily be seen from Fig. 3, and their field distributions may be found from (139) and (140). The relations between the various amplitudes $C_1^{(q)}$, $C_2^{(q)}$, $C_3^{(q)}$, $D_1^{(q)}$, etc., may also be determined from the boundary conditions (see Fig. 4).

TERMINAL AND INPUT IMPEDANCE

The terminal impedances of a wave guide containing several sections of dielectric material at one end become at once, from (23), (24), (31), (35), (143), and (143a) applied to (141) and (142),

$$Z_s^{(q)} = \left\{ \begin{matrix} -iK_2^{(1)}/\omega\epsilon_2^* \\ i\omega\mu_2/K_2^{(2)} \end{matrix} \right\} \bar{\Gamma}^{(q)}, \quad q=1, 2, \quad (146)$$

where the upper expression in the brackets is to be used when $q=1$ and the lower when $q=2$. From the general relations (39) and (40), $\rho_s^{(q)}$ and $\Phi_s^{(q)}$ of our termination are defined and may be obtained from (43) and (44). In a similar fashion $Z_{in}^{(q)}$ may be obtained from (49) and (50). (See Figs. 5 and 6.)

When the dielectrics are perfect, we have

$$K_j^{(q)} = i\beta_j^{(q)}, \quad (j=1, 2, 3), \quad \tau_2^{(1)} = \epsilon_2\beta_3^{(1)}/\epsilon_3\beta_2^{(1)}; \quad \tau_2^{(2)} = \mu_3\beta_2^{(2)}/\mu_2\beta_3^{(2)}. \quad (147)$$

Then (146) becomes

$$Z_s^{(q)} = \left\{ \begin{matrix} i\beta_2^{(1)}/\omega\epsilon_2 \\ i\omega_2/\beta_2^{(2)} \end{matrix} \right\} \times \left\{ \frac{\tan \beta_2^{(q)} l_2 + \tau_2^{(q)} \tan \beta_3^{(q)} l_3}{1 - \tau_2^{(q)} \tan \beta_2^{(q)} l_2 \tan \beta_3^{(q)} l_3} \right\}. \quad (148)$$

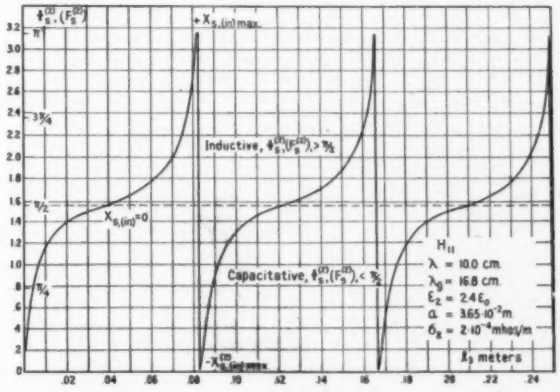


FIG. 6. The $\Phi_s^{(2)}$ function for the termination of Fig. 4. This also represents $F_s^{(2)}$ for the hollow pipe of length l_3 terminated in a dielectric section l_2.

Equation (148) is a pure reactance, as one would expect, since there are now no dissipative elements in the termination when the dielectric conductivities are zero. However, the general expression (146) is complex and has a real or resistive term $R_s^{(q)}$ which is positive and non-vanishing. When

$$A_s^{(q)2} \ll 1, \quad \phi_c \left\{ \begin{matrix} e \\ m \end{matrix} \right\}^2 \ll 1, \tag{149}$$

which is characteristic of a guide with low overall attenuation, low attenuation per unit length, and negligible leakage conductance, the extreme values of $R_s^{(q)}$ and $X_s^{(q)}$ become, upon simplification of (41), (42), (49), (50),

$$(R_s^{(q)})_{\min} = R_c \left\{ \begin{matrix} e \\ m \end{matrix} \right\} A_{s,2}^{(q)}, \tag{150}$$

at

$$\tan (\beta_3^{(q)} l_3 + \Phi_{s,2}^{(q)}) = \beta_3^{(q)} A_{s,2}^{(q)} / \alpha_3^{(q)}, \tag{150a}$$

$$(R_s^{(q)})_{\max} = R_c \left\{ \begin{matrix} e \\ m \end{matrix} \right\} / A_{s,2}^{(q)}, \tag{151}$$

at

$$\beta_3^{(q)} l_3 = n\pi - \Phi_{s,2}^{(q)}, \tag{151a}$$

$$(X_s^{(q)})_{\max} = [R_c \left\{ \begin{matrix} e \\ m \end{matrix} \right\} / 2A_{s,2}^{(q)}](1 - \alpha_3^{(q)} / \beta_3^{(q)}), \tag{152}$$

at

$$\beta_3^{(q)} l_3 = (n\pi - \Phi_{s,2}^{(q)})(1 - \alpha_3^{(q)} / \beta_3^{(q)}) - \rho_{s,2}^{(q)}, \tag{152a}$$

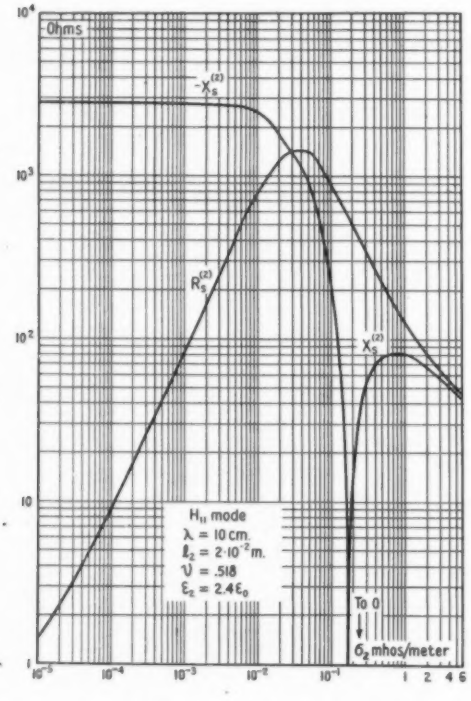


FIG. 7. The terminal impedance at $\lambda_g = 16.8$ cm, H_{11} mode, of a section of the pipe of Fig. 4 containing a 2-cm length of dielectric, as a function of dielectric conductivity σ_2, $l_3 = 0$.

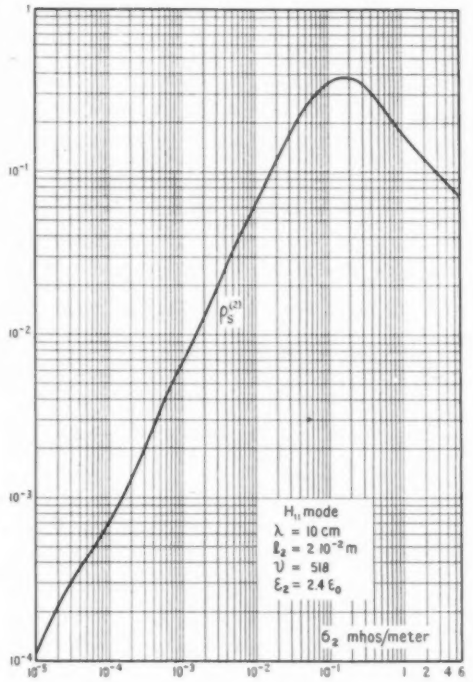


FIG. 8. $\rho_s^{(2)}$ for the termination of Fig. 7 as a function of dielectric conductivity σ_2.

$$(X_s^{(q)})_{\min} = -[R_c \left\{ \begin{matrix} e \\ m \end{matrix} \right\} / 2A_{s,2}^{(q)}] \times (1 + \alpha_3^{(q)} / \beta_3^{(q)}), \tag{153}$$

at

$$\beta_3^{(q)} l_3 = (n\pi - \Phi_{s,2}^{(q)})(1 + \alpha_3^{(q)} / \beta_3^{(q)}) + \rho_{s,2}^{(q)}, \tag{153a}$$

where

$$A_{s,2}^{(q)} = \alpha_3^{(q)} l_3 + \rho_{s,2}^{(q)}. \tag{154}$$

The functions $\rho_{s,2}^{(q)}$ and $\Phi_{s,2}^{(q)}$ are obtained from (43), (44), and (146), with $l_3 = 0$. For the guide terminated in $Z_s^{(q)}$, (146), we may simply obtain the extreme values of input resistance and reactance by replacing the subscript s on $R_s^{(q)}$ and $X_s^{(q)}$ by in and changing l_3 to s in the above formulae, as well as $\alpha_3^{(q)}$, $\beta_3^{(q)}$ to $\alpha_1^{(q)}$, $\beta_1^{(q)}$, respectively. The quantity $A_{s,2}^{(q)}$ becomes $A_s^{(q)}$, and is given by (50), where $\rho_s^{(q)}$ and $\Phi_s^{(q)}$ are obtained from (43) and (44), for the impedance of the two dielectric termination (146). Figure 4 illustrates the variation of this impedance with l_3, when medium three is air. As can be seen from Fig. 4, and (151)–(153), the terminal (or input) resistances and reactances approach critical values of approximately half a megohm, which are the same order of magnitude as the values

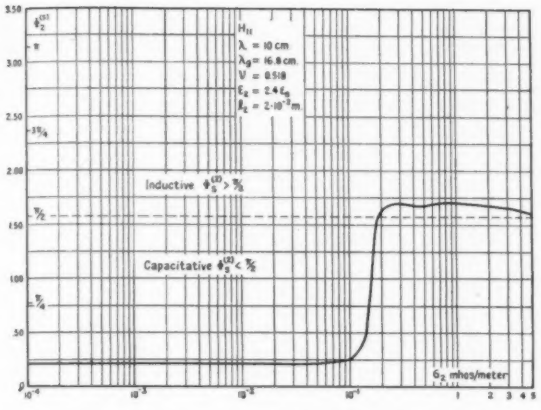


FIG. 9. $\Phi_s^{(2)}$ for the termination of Fig. 7 as a function of the dielectric conductivity σ_2.

obtained in the case of the conventional two-wire or coaxial line.* Figures 5 and 6 are calculated from (43) and (44) and give the variation of the terminal functions $\rho_s^{(2)}$ and $\Phi_s^{(2)}$ with l_3. The phase function $\Phi_s^{(2)}$ varies between $\pi-\delta$ and δ ($\delta=10^{-3}$) and changes very abruptly in the immediate neighborhood of anti-resonance. This change would be discontinuous for perfect media, and $\rho_s^{(2)}$ would vanish. For the low loss termination or circuit, $\rho_s^{(2)}$ (Fig. 5) varies over a fairly wide range and reaches a maximum value at anti-resonance, but is never at any time greater than about $1.5\cdot10^{-3}$ in this instance (see also Fig. 7).

For the important special case of a one-medium termination (146) reduces to

$$Z_s^{(q)} = \left\{ \begin{matrix} -iK_2^{(1)}/\omega\epsilon_2^* \\ i\omega\mu_2/K_2^{(2)} \end{matrix} \right\} \tanh K_2^{(q)} l_2, \quad l_3=0. \quad (155)$$

When the dielectrics are perfect this becomes

$$Z_s^{(q)} = \left\{ \begin{matrix} i\beta_2^{(1)}/\omega\epsilon_2 \\ i\omega\mu_2/\beta_2^{(2)} \end{matrix} \right\} \tan \beta_2^{(q)} l_2, \quad (156)$$

which like (148) is a pure reactance. Figure 7 illustrates the variation of $Z_s^{(2)}$ of the one dielectric case (155) as the conductivity of the medium is varied.[1] Antiresonance effects are noticeable when σ_2 reaches a certain value; a brief inspection of (155) indicates that this is quite possible, for instead of changing l_3 (or l_2), as was done for Fig. 4, $K_2^{(2)}$ is now varied. Figures 8 and 9 show $\rho_s^{(2)}$ and $\Phi_s^{(2)}$ as functions of σ_2. When $\sigma_2>0.18$ mho/m, $\epsilon=2.4\epsilon_0$, the termination is inductive and in effect lengthens the guide, though not greatly; for $\sigma_2<0.18$ it is capacitative and shortens the pipe. Figure 10 illustrates the effect of conductivity on the attenuation and phase shift constants of the medium [a constant attenuation for the metal guide is included, additively when (77) permits; otherwise (76) is used]. The phase shift $\beta_3^{(2)}$ is affected much less critically than $\alpha_3^{(2)}$ when $\sigma_2<10^{-1}$ mho/m.

BEHAVIOR OF THE GUIDE AS A RESONATOR[2]

IX

The theory of the previous sections applies in general for the hollow pipe as a guide or resonator. For resonator behavior the source is loosely coupled to the hollow cavity, and we may treat its distribution as being essentially plane,

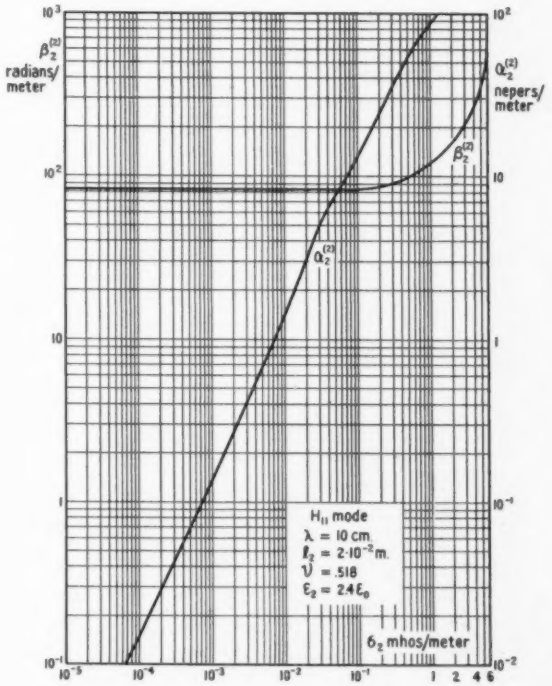


FIG. 10. The attenuation $\alpha_2^{(2)}$ and phase shift $\beta_2^{(2)}$ of the pipe of Fig. 7 as a function of dielectric conductivity σ_2.

*See Part I, reference 1, Figs. 2 and 3.

[1] Kennelley's Chart Atlas is helpful when dealing with hyperbolic functions of complex variables.

[2] D. Middleton, Phys. Rev. **63**, 343 (1943), gives a discussion of cylindrical cavity resonators containing several perfect dielectrics.

as was pointed out in Section III. If the coupling is sufficiently loose, the frequency of the source for which the hollow pipe becomes resonant, i.e., when standing waves are present in the interior, is the natural frequency of the cavity. Equation (62) yields the condition for resonance. The Q of the circuit,[3] consisting of source, detector, and closed hollow pipe, must be high enough so that (63) is satisfied. When the standing wave patterns become very broad we are no longer able to determine the maximum amplitudes accurately, and in fact the term resonance becomes meaningless. Analytically $\Phi_0^{(q)}$ and $\Phi_s^{(q)}$, as well as $\rho_0^{(q)}$ and $\rho_s^{(q)}$, may be quite impossible to determine, but they can always be obtained experimentally, as explained in Section VII (d) and illustrated in (e). The fields and charge and current distributions in the resonant case may then be determined accurately from (58) and (59), or their amplitudes from (60) and (61), with the help of (54), except in the immediate vicinity of the source and detector, as was pointed out in somewhat greater detail in Section III.

The writer wishes to express his appreciation to Professor Ronold King for his encouragement and many valuable suggestions in the preparation of this manuscript, and for the data of Figs. 1 and 2.

[3] Part I, reference 10, Chapter 10.

An Approximate Representation of the Electromagnetic Field in the Vicinity of a Symmetrical Radiator

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The actual quadrature components of current along a symmetrical center-driven antenna of non-vanishing radius are represented by simple distribution functions. The magnetic and electric vectors in the near zone are then determined by carrying out the calculus of the field theory involved. The resulting expressions, while more intricate than existing formulas based on the assumption that the current in any antenna is sinusoidally distributed, represent approximately the field conditions about antennas used in practice.

THE electromagnetic field at points outside an antenna of half-length h and radius a, and oriented as shown in Fig. 1, may be conveniently calculated from the vector potential given by

$$A_z=\frac{\Pi}{4\pi}\int_{-h}^{+h} I_{z'}\frac{e^{-j\beta R}}{R}dz'. \tag{1}$$

Here the universal magnetic constant $\Pi=4\pi\times10^{-7}$ henry per meter. $I_{z'}$ is the complex current amplitude in amperes flowing in the element dz'. β is the propagation constant in radians per meter, and equals $2\pi/\lambda$, where λ is the wave-length. R is the distance from the point (r, θ, z) where the vector potential is computed to the element dz'. That is,

$$R=[(z-z')^2+r^2]^{\frac{1}{2}}. \tag{2}$$

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The electric and magnetic field vectors in empty space may be computed directly from (1) using the relations

$$B=\text{curl } A \tag{3}$$

and

$$E=1/j\omega\Pi\Delta \text{ curl curl } A=-j(c^2/\omega)\text{ curl } B. \tag{4}$$

In terms of the universal electric constant $\Delta=8.85\times10^{-12}$ farad per meter, and the magnetic constant Π,

$$c=1/(\Pi\Delta)^{\frac{1}{2}}=3\times10^8 \text{ meters per second}, \tag{5}$$

$$\beta=\omega/c. \tag{6}$$

The components of the curl in cylindrical coordinates involving the z component of A are:

$$\begin{aligned}&\text{curl}_r A=(1/r)(\gamma A_z/\gamma\theta),\\ &\text{curl}_\theta A=-\gamma A_z/\gamma r,\\ &\text{curl}_z A=0.\end{aligned} \tag{7}$$

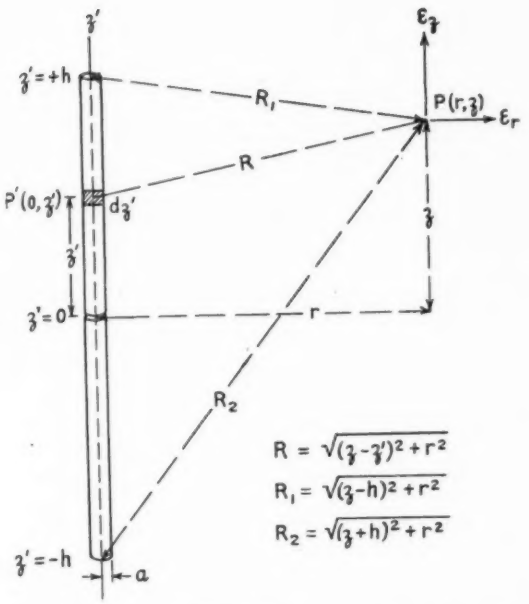

FIG. 1. Illustrates orientation of antenna relative to the primed and unprimed coordinate systems.

Since rotational symmetry obtains, $\gamma A_z/\gamma\theta=0$, and accordingly (3) becomes

$$B_\theta=\text{curl}_\theta A=-\gamma A_z/\gamma r. \tag{8}$$

The components of the curl in cylindrical coordinates involving the θ component of B are

$$\begin{aligned}&\text{curl}_r B=-\gamma B_\theta/\gamma z,\\ &\text{curl}_\theta B=0,\\ &\text{curl}_z B=(1/r)(\gamma/\gamma r)(rB_\theta).\end{aligned} \tag{9}$$

Therefore, the radial component of the electric field vector is

$$E_r=(jc^2/\omega)(\gamma B_\theta/\gamma z) \tag{10}$$

and the tangential component is

$$E_z=(-jc^2/\omega)(1/r)(\gamma/\gamma r)(rB_\theta). \tag{11}$$

From a theoretical point of view the calculation of the field vectors in the vicinity of an antenna is a relatively simple problem, if the distribution of current is known. However, the exceedingly intricate form of the actual current distribution in antennas of finite radius seems to preclude a direct determination of the field. An alternative approach, which is probably satisfactory in practice, is to represent the quadrature components of current along the antenna by simple distribution functions. This is actually possible to a good degree of approximation.

For the component of current in-phase with the driving potential V_0^e one writes

$$I_0''\left(\frac{\cos\beta z'-\cos\beta h}{1-\cos\beta h}\right), \tag{12}$$

and for the component of current in quadrature with V_0^e the expression is

$$I_0'\frac{\sin\beta(h-|z'|)}{\sin\beta h}. \tag{13}$$

In (12) and (13), I_0'' and I_0' are the current components at the input terminals of the antenna. Their values may be ascertained directly from the relation

$$I_0=V_0^eY_{00}, \tag{14}$$

where I_0 is the complex current at the driving point, and Y_{00} is the input admittance of the antenna in question.

Naturally there are intrinsic difficulties in any representation of the current except the correct one. The reader is referred to the literature for a discussion of this point.[1,2]

Substituting (12) in (1)

$$A_z''=\frac{\Pi I_0''}{4\pi(1-\cos\beta h)}\left\{\frac{1}{2}\int_{-h}^{+h}\frac{e^{-j\beta(R-z')}}{R}dz'+\frac{1}{2}\int_{-h}^{+h}\frac{e^{-j\beta(R+z')}}{R}dz'-\cos\beta h\int_{-h}^{+h}\frac{e^{-j\beta R}}{R}dz'\right\}. \tag{15}$$

Using (8) one has

$$\begin{aligned}B_\theta''=\frac{-\Pi I_0''}{4\pi(1-\cos\beta h)}&\times\left\{\tfrac{1}{2}e^{+j\beta z}\frac{\gamma}{\gamma r}\int_{j\beta(R_1-h+z)}^{j\beta(R_2+h+z)}\frac{e^{-u}}{u}du\right.\\&+\tfrac{1}{2}e^{-j\beta z}\frac{\gamma}{\gamma r}\int_{j\beta(R_2-h-z)}^{j\beta(R_1+h-z)}\frac{e^{-v}}{v}dv\\&\left.-\cos\beta h\int_{-h}^{+h}\frac{\gamma}{\gamma r}\frac{e^{-j\beta R}}{R}dz'\right\}.\end{aligned} \tag{16}$$

In (16)

$$R_1=[(z-h)^2+r^2]^{\frac{1}{2}}, \tag{17}$$

$$R_2=[(z+h)^2+r^2]^{\frac{1}{2}}. \tag{18}$$

Upon noting that

$$\frac{\gamma}{\gamma r}\int_{w_1}^{w_2}\frac{e^{-w}}{w}dw=\frac{e^{-w_2}}{w_2}\frac{\gamma w_2}{\gamma r}-\frac{e^{-w_1}}{w_1}\frac{\gamma w_1}{\gamma r} \tag{19}$$

and carrying out the required differentiation,

[1] Ronold King and Charles W. Harrison, Jr., Proc. I. R. E. **31**, 548–567 (1943).

[2] Charles W. Harrison, Jr., and Ronold King, Proc. I. R. E. **31**, 693–697 (1943).

$$B_\theta'' = \frac{\Pi I_0''}{4\pi(1-\cos\beta h)r} \times \left\{ (jR_1 \sin\beta h + (h-z)\cos\beta h)\frac{e^{-j\beta R_1}}{R_1} + (jR_2 \sin\beta h + (h+z)\cos\beta h)\frac{e^{-j\beta R_2}}{R_2} - r^2 \cos\beta h \left(j\beta \int_{-h}^{+h} \frac{e^{-j\beta R}}{R^2} dz' + \int_{-h}^{+h} \frac{e^{-j\beta R}}{R^3} dz' \right) \right\}. \quad (20)$$

The integral

$$I = \int_{-h}^{+h} \frac{e^{-j\beta R}}{R^n} dz' \quad (21)$$

with $n=1$ plays a major role in the solution of the problem of coupled antennas.[3] The evaluation of (21) may be readily carried out, using numerical methods.

From (13), (1), and (8)

$$B_\theta' = \frac{j\Pi I_0'}{4\pi r \sin\beta h}\{2\cos\beta h e^{-j\beta R_0} - e^{-j\beta R_1} - e^{-j\beta R_2}\}, \quad (22)$$

$$R_0 = (z^2+r^2)^{\frac{1}{2}}. \quad (23)$$

Equations (20) and (22), when appropriately combined in quadrature according to (14), completely define the magnetic field B_θ about a linear radiator of finite radius.

The radial electric field is readily obtained from (20) and (22) by applying (10):

$$E_r'' = \frac{jI_0''}{4\pi\omega\Delta r(1-\cos\beta h)} \times \left\{ \left(\beta(h+z)\sin\beta h + \frac{r^2\cos\beta h}{R_2^2} - j\frac{\beta(h+z)^2\cos\beta h}{R_2} \right)\frac{e^{-j\beta R_2}}{R_2} - \left(\beta(h-z)\sin\beta h + \frac{r^2\cos\beta h}{R_1^2} - j\frac{\beta(h-z)^2\cos\beta h}{R_1} \right)\frac{e^{-j\beta R_1}}{R_1} - r\cos\beta h \left[3\left(\frac{e^{-j\beta R_2}}{R_2^3} - \frac{e^{-j\beta R_1}}{R_1^3} \right) + \beta^2\left(\frac{e^{-j\beta R_1}}{R_1} - \frac{e^{-j\beta R_2}}{R_2} \right) + j\beta^3\left(Ci\beta R_1 - jSi\beta R_1 - Ci\beta R_2 + jSi\beta R_2 \right) \right] \right\}; \quad (24)$$

$$E_r' = \frac{jI_0'}{4\pi c\Delta r \sin\beta h}\left\{ \frac{(z-h)}{R_1}e^{-j\beta R_1} + \frac{(z+h)}{R_2}e^{-j\beta R_2} - \frac{2z\cos\beta h}{R_0}e^{-j\beta R_0} \right\}. \quad (25)$$

Using (20) and (22) with (11), the tangential electric field is found to be:

$$E_z'' = \frac{-jI_0''}{4\pi\omega\Delta(1-\cos\beta h)} \left\{ \left(\beta\sin\beta h - \frac{(h-z)(1+j\beta R_1)\cos\beta h}{R_1^2} \right)\frac{e^{-j\beta R_1}}{R_1} + \left(\beta\sin\beta h - \frac{(h+z)(1+j\beta R_2)\cos\beta h}{R_2^2} \right) \times \frac{e^{-j\beta R_2}}{R_2} - \cos\beta h \left(j2\beta \int_{-h}^{+h} \frac{e^{-j\beta R}}{R^2} dz' + (2+\beta^2 r^2)\int_{-h}^{+h} \frac{e^{-j\beta R}}{R^3} dz' - 3r^2\left(j\beta \int_{-h}^{+h} \frac{e^{-j\beta R}}{R^4} dz' + \int_{-h}^{+h} \frac{e^{-j\beta R}}{R^5} dz' \right) \right) \right\}; \quad (26)$$

$$E_z' = \frac{jI_0'}{4\pi c\Delta}\left\{ 2\cos\beta h \frac{e^{-j\beta R_0}}{R_0} - \frac{e^{-j\beta R_1}}{R_1} - \frac{e^{-j\beta R_2}}{R_2} \right\}. \quad (27)$$

The local radial field about an antenna of non-vanishing radius is given by (24) and (25) combined in quadrature in accordance with (14). The electric field in the z direction is given by a similar combination of (26) and (27). All field expressions apply to the case of an isolated symmetrical center-driven antenna of half-length h and radius a, or to antennas of full length h and radius a erected vertically over a perfectly conducting half-space of infinite extent.

The general complexity of the relations given here precludes a general graphical portrayal of field conditions about antennas of specified dimensions.

The opinions contained herein are those of the writer, and do not reflect the views of the Navy Department or the Naval Service.

[3] Ronold King and Charles W. Harrison, Jr., J. App. Phys. **15**, 170 (1944).

Theory of Membranes in Shape of Developable Surfaces

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THE theory of equilibrium developed hereafter is an application to developable surfaces of the general equations of equilibrium of membranes; such theory disregards local bending in the thickness of the membrane. The following analysis deals with general developables, exclusive of cones and cylinders.

I. GEOMETRY OF DEVELOPABLE SURFACES

A developable surface can be generated by the tangents to a skew curve, which is called the edge of regression of the developable. The parametric equations of the skew curve are taken under the form

$$x_1=f_1(\lambda_1);\quad x_2=f_2(\lambda_1);\quad x_3=f_3(\lambda_1). \tag{1}$$

Thus, the equations of the developable can be taken under the form

$$\begin{aligned} x_1&=f_1(\lambda_1)+\lambda_2 f_1'(\lambda_1),\\ x_2&=f_2(\lambda_1)+\lambda_2 f_2'(\lambda_1),\\ x_3&=f_3(\lambda_1)+\lambda_2 f_3'(\lambda_1); \end{aligned} \tag{2}$$

in which the $'$ indicates the derivation. When λ_1 represents the length of the arc along the edge of regression, it is more convenient to replace λ_2, which is the length of the tangent between the point of contact with the edge of regression and the point under consideration, by $\lambda_2-\lambda_1$. Thus, λ_2 represents the distance between the point under consideration and a fixed point P on the edge of regression, distance taken successively along the tangent, and along the edge of regression. Then:

$$\begin{aligned} x_1&=f_1(\lambda_1)+(\lambda_2-\lambda_1)f_1'(\lambda_1),\\ x_2&=f_2(\lambda_1)+(\lambda_2-\lambda_1)f_2'(\lambda_1),\\ x_3&=f_3(\lambda_1)+(\lambda_2-\lambda_1)f_3'(\lambda_1). \end{aligned} \tag{3}$$

With this representation, the curves λ_1 are orthogonal to the generators λ_2, as the condition of orthogonality which is

$$\sum_m \frac{\partial x_m}{\partial\lambda_1}\frac{\partial x_m}{\partial\lambda_2}=0$$

becomes

$$(\lambda_2-\lambda_1)(f_1'f_1''+f_2'f_2''+f_3'f_3'')=0.$$

This condition is easily verified since λ_1 is the arc of the edge of regression, thus:

$$f_1'^2+f_2'^2+f_3^2=1,$$

which, by derivation, gives the condition above.

As the generators of a developable surface are lines of curvature, the lines λ_1, orthogonal to the generators, constitute the second set of lines of curvature.

We call ρ the radius of curvature of the edge of regression

$$1/\rho=(f_1''^2+f_2''^2+f_3''^2)^{\frac{1}{2}}. \tag{4}$$

The metric elements of the surface, i.e., the ratios:

$$\epsilon_1=ds_1:d\lambda_1;\quad \epsilon_2=ds_2:d\lambda_2$$

in which s_1, s_2 are the arcs along the curves-coordinates, and also the direction cosines L_1, L_2, L_3 of the normal to the surface are equal to:

$$\epsilon_1=(f_1''^2+f_2''^2+f_3''^2)^{\frac{1}{2}}(\lambda_2-\lambda_1)=\frac{\lambda_2-\lambda_1}{\rho};\quad \epsilon_2=1;$$

$$\begin{aligned} L_1&=-\rho(f_2''f_3'-f_3''f_2');\\ L_2&=-\rho(f_3''f_1'-f_1''f_3');\\ L_3&=-\rho(f_1''f_2'-f_2''f_1'). \end{aligned} \tag{5}$$

Also:

$$\frac{\partial^2\chi_m}{\partial\lambda_1^2}=(\lambda_2-\lambda_1)f_m'''-f_m'';$$

$$\frac{\partial^2\chi_m}{\partial\lambda_1\partial\lambda_2}=f_m'';\quad \frac{\partial^2\chi_m}{\partial\lambda_2^2}=0.$$

The geodesic curvatures of the curves λ_1, λ_2 are, as it appears from Fig. 1,

$$\frac{1}{r_1}=\frac{1}{\lambda_2-\lambda_1};\quad \frac{1}{r_2}=0. \tag{6}$$

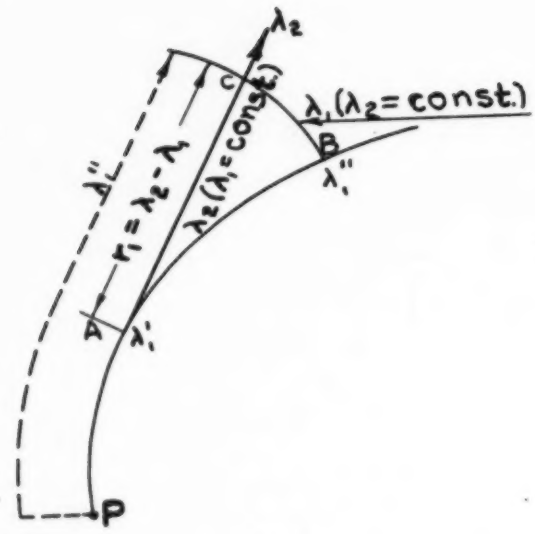


FIG. 1.

The principal curvatures of the surface are:

$$\frac{1}{R'}=-\frac{\sum f_1'''(f_2''f_3'-f_3''f_2')}{(\lambda_2-\lambda_1)(\sum f_1''^2)^{\frac{1}{2}}}=\frac{\begin{vmatrix} f_1''' & f_2''' & f_3''' \\ f_1'' & f_2'' & f_3'' \\ f_1' & f_1' & f_3' \end{vmatrix}\rho^3}{(\lambda_2-\lambda_1)};$$
$$\frac{1}{R''}=0. \tag{7}$$

It is to be noted that the curvature $1:R'$ is simply related to the torsion T of the edge of regression

$$\frac{1}{R'}=\frac{T}{\epsilon_1}=\frac{T\rho}{r_1}. \tag{8}$$

II. EQUATIONS OF EQUILIBRIUM

The general equations of equilibrium are written between the applied surface loads and the stress flows. The three components of the applied loads per unit surface are: Z, along the normal to the surface; X_1, X_2 along the tangents to the curves-coordinates. The three components of stress flow (i.e., stress multiplied by thickness) are n_1, n_2 normal flows along the curves-coordinates, n_{12} shear flow along the curves-coordinates. The three equations of equilibrium represent one, the equilibrium along the normal; the two others, the equilibrium in the tangent plane—the two latter being merely the equations of equilibrium in polar coordinates, the pole being taken on the edge of regression

$$\begin{gathered}\frac{n_1}{R'}+Z=0,\\ \frac{1}{\epsilon_1}\frac{\partial n_1}{\partial\lambda_1}+\frac{1}{(\lambda_2-\lambda_1)^2}\frac{\partial[n_{12}(\lambda_2-\lambda_1)^2]}{\partial\lambda_2}+X_1=0,\\ \frac{1}{\lambda_2-\lambda_1}\frac{\partial[n_2(\lambda_2-\lambda_1)]}{\partial\lambda_2}+\frac{1}{\epsilon_1}\frac{\partial n_{12}}{\partial\lambda_1}\\ -\frac{n_1}{\lambda_2-\lambda_1}+X_2=0.\end{gathered} \tag{9}$$

It appears that n_{12} and n_2 can be calculated by successive integration:

$$\begin{aligned} n_1&=-ZR',\\ n_{12}&=-\frac{1}{(\lambda_2-\lambda_1)^2}\int_{\lambda_1}^{\lambda_2}(\lambda_2-\lambda_1)^2\\ &\quad\times\left[X_1+\frac{\partial n_1}{\partial\lambda_1}\frac{1}{\epsilon_1}\right]d\lambda_2+\frac{C(\lambda_1)}{(\lambda_2-\lambda_1)^2}\\ &=n_{12}^0+\frac{C(\lambda_1)}{(\lambda_2-\lambda_1)^2},\\ n_2&=\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\left[n_1-(\lambda_2-\lambda_1)X_2\right.\\ &\quad\left.-\rho\frac{\partial n_{12}}{\partial\lambda_1}\right]d\lambda_2+\frac{D(\lambda_1)}{\lambda_2-\lambda_1}\\ &=n_2^0+\frac{\rho C'(\lambda_1)}{(\lambda_2-\lambda_1)^2}+\frac{\rho C(\lambda_1)}{(\lambda_2-\lambda_1)^3}+\frac{D(\lambda_1)}{\lambda_2-\lambda_1}. \end{aligned} \tag{10}$$

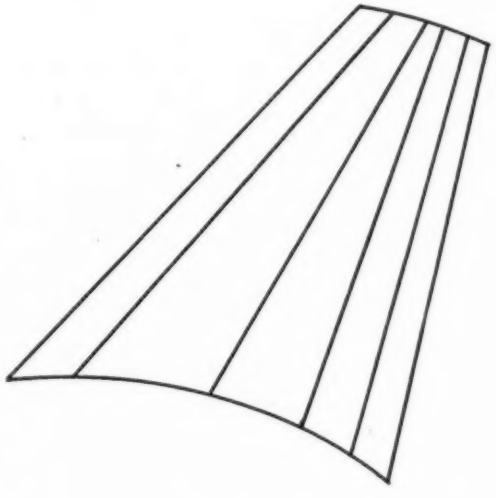


FIG. 2.

The two successive integrations introduce two arbitrary functions $C(\lambda_1)$ and $D(\lambda_1)$ which are to be determined by the conditions at the edges of the surface. n_{12}^0 and n_2^0 are the values of n_{12} and n_2 when the functions C and D are equal to zero, the limits of integration being those indicated. In the case when the applied loads are constant in value and direction, along each generator, or more generally in the case when the loads are functions of a power p of $\lambda_2-\lambda_1$, i.e., of the distance to the edge of regression measured along the generator, the stress flows have the following values in which we omit C and D:

$$
\begin{aligned}
&Z=Z_p(\lambda_2-\lambda_1)^p; \quad X_1=X_{1p}(\lambda_2-\lambda_1)^p;\\
&X_2=X_{2p}(\lambda_2-\lambda_1)^p; \quad n_1=(\lambda_2-\lambda_1)^{p+1}\nu_1;\\
&\nu_1=-\frac{Z_p(\sum f_1''^2)^{\frac{1}{2}}}{\sum f_1'''(f_2''f_3'-f_3''f_2')};\\
&n_{12}=(\lambda_2-\lambda_1)^{p+1}\nu_{12}+(\lambda_2-\lambda_1)^p\rho\nu_1\frac{p+1}{p+2};\\
&\nu_{12}=-\frac{1}{p+3}\left[X_{1p}+\rho\frac{\partial\nu_1}{\partial\lambda_1}\right];\\
&n_2=(\lambda_2-\lambda_1)^{p+1}\nu_2+(\lambda_2-\lambda_1)^p\bar{\nu}_2\\
&\qquad\qquad+(\lambda_2-\lambda_1)^{p-1}\rho^2\nu_1\frac{p+1}{p+2};\\
&\nu_2=-X_{2p}+\nu_1-\rho\frac{\partial\nu_{12}}{\partial\lambda_1}; \quad \bar{\nu}_2=\rho\nu_{12}-\frac{\rho^2}{p+1}\frac{\partial\nu_1}{\partial\lambda_1}.
\end{aligned}
\tag{11}
$$

If $p=-1$, all three stress flows are limited to their first factor, and if $p=0$, n_2 is limited to its two first factors.

The most important cases correspond to $p=0$, in which case the surface loads are constant along each generator, and to $p=-1$, in which case the surface loads are inversely proportional to the distance to the edge of regression, for instance in the case of a membrane hyperbolically reinforced towards the edge of regression.

If C and D are zero, in the case $p=0$, the stress flows are zero along the edge of regression, and in the case $p=-1$, the stress flows are finite along the edge. The last hypothesis will not be examined; assuming that the surface loads are finite in every point, it appears that the stress flows along the edge of regression are determined

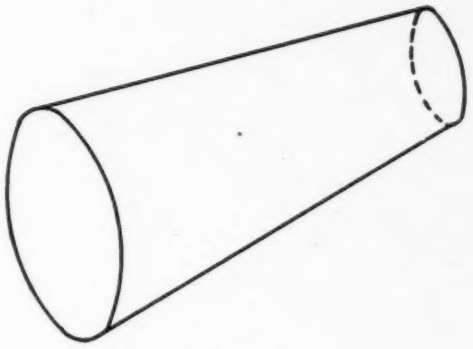


FIG. 3.

by the factors C and D, i.e.:

$$
\begin{aligned}
&n_1=0; \quad n_{12}=\frac{C(\lambda_1)}{(\lambda_2-\lambda_1)^2};\\
&n_2=\frac{\rho C'(\lambda_1)}{(\lambda_2-\lambda_1)^2}+\frac{\rho C(\lambda_1)}{(\lambda_2-\lambda_1)^3}+\frac{D(\lambda_1)}{\lambda_2-\lambda_1}.
\end{aligned}
\tag{12}
$$

The generators being tangent to the edge of regression, it results that the stress flows n_2 do not appear as forces perpendicular to the edge of regression, but as shear flow, the value of which is, from Fig. 1, in which the triangle ABC is considered as infinitely small:

$$
\begin{aligned}
\bar{n}_{12}&=-\operatorname{limit}\left[\frac{n_2\cdot\overline{CB}}{\overline{AB}}\right]\\
&=-\lim_{\lambda_2=\lambda_1}\left[\frac{n_2(\lambda_2-\lambda_1)}{\rho}\right]=-\frac{D(\lambda_1)}{\rho}.
\end{aligned}
\tag{13}
$$

If D is different from zero, and C is equal to zero, there is a finite shear flow along the edge of regression, and more generally, the shear flow has the value:

$$
\bar{n}_{12}=n_{12}-\lim_{\lambda_2=\lambda_1}\left[\frac{1}{\rho}n_2(\lambda_2-\lambda_1)\right]. \tag{14}
$$

If C is not zero, there is an infinite shear flow along the edge of regression, and such a case has no practical interest.

III. EDGE CONDITIONS

From the results established above, the conditions of statically determined equilibrium of the membrane can be deduced.

Case I. The membrane does not include its edge of regression.

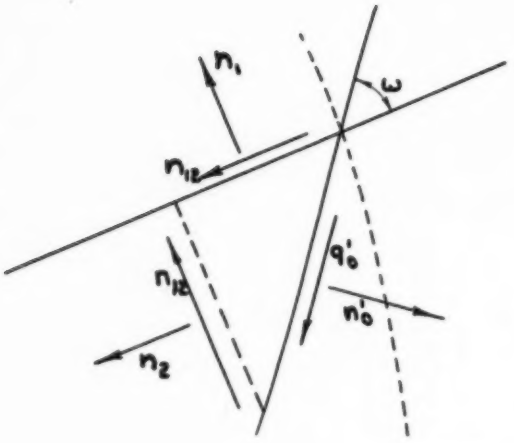


FIG. 4.

(IA). The membrane is limited by two generators and two lines of curvature (Fig. 2).

If two conditions are imposed at each generator, these two conditions give two relations between the unknown functions $C(\lambda_1)$ and $D(\lambda_1)$. Thus the arbitrary functions are determined. More exactly, if the shear is not given, $C(\lambda_1)$ appears through a differential equation of first degree and it is necessary to give the shear along one generator, or a general resultant of shear along one of the two limiting curves, thus giving the possibility of determining the arbitrary constant which appears in $C(\lambda_1)$. Moreover, it is impossible to impose the value of shear at both ends of a generator, because this would give two equations for $c(\lambda_1)$ and zero equation for $D(\lambda_1)$. Thus, at the two ends of each generator, it is possible to impose one of the following groups of conditions: (a) normal flow determined at both ends of each generator, and a resultant of shear determined along one edge; (b) shear and normal flow determined at one end of each generator; (c) shear flow at one end of each generator and normal flow at the other end.

Moreover, along a generator which is the edge of the membrane, the shear and normal flows are determined. Thus a generator which coincides with an edge must be reinforced by a member accumulating or redistributing the shear flow, and also by a beam absorbing the normal flow. This beam can be suppressed when there are no distributed loads on the membrane, or at least in the region of the membrane near the edge. It may also be possible to use a part of the membrane along the generator, reinforced if needed, to constitute a beam.

(IB). The membrane is closed, i.e., limited to two lines of curvature (Fig. 3). If the shear is given at one end with the normal flow given at same or opposite and of each generator (Cases IAb and IAc), the membrane will be in equilibrium.

If the normal forces are given at both ends (Case IAa), it is necessary to verify if the shear flow function will be univalued, i.e., if by integration of Eqs. (9), n_{12} will be the same after a complete integration along a closed curve. If not, this will mean that the membrane will not be in equilibrium under the given loads, and will need one or several members along one or several generators to absorb the difference of shear flow on both sides of the member. If such members exist, or if the loads at the end curves are in equilibrium with the applied loads, the calculation will be made as for the Case IA.

(IC). The membrane is open or closed, the lines of curvature are replaced by two curves which do not coincide with the generators. The conditions in IA and IB are not essentially changed, the only modification consists in the fact that the end shear and normal flow are not the quantities n_{12} and n_2, but combinations of n_1, n_{12}, and n_2. The shear q_0' and normal flow n_0' along

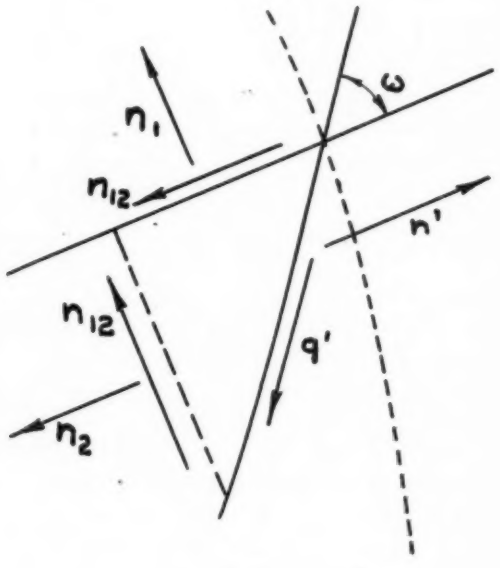


FIG. 5.

the edge (Fig. 4) will be:

$$n_0' = n_1 \cos^2 \omega + n_2 \sin^2 \omega + 2n_{12} \sin \omega \cos \omega,$$
$$q_0' = (n_1 - n_2) \sin \omega \cos \omega + n_{12}(\sin^2 \omega - \cos^2 \omega) \quad (15)$$

(positive signs as shown on Fig. 4).

The shear and normal flow can be replaced by the flow along the generators and the "generalized" shear conjugate to the direction of the edge and the generators. Calling these flows n' and q' we find (Fig. 5):

$$n' = \frac{n_0'}{\sin \omega} = n_1 \cos \omega \cot \omega + n_2 \sin \omega + 2n_{12} \cos \omega, \quad (16)$$
$$q' = q_0' + n_0' \cot \omega = n_{12} + n_1 \cot \omega.$$

In this manner it is found that, provided it is made use of conjugate flow along the edge, the conjugate shear is not a function of n_2; thus it does not contain C' and D, and the results obtained under IA and IB remain unchanged, as the knowledge of q is equivalent to the knowledge of n_{12} (n_1 is determined in all cases: $n_1 = -ZR'$).

On the contrary, if the end conditions are given in terms of shear and normal flows q and n, a condition involving one of these two quantities only does not give the possibility to determine C directly. Thus: (a) Conditions introducing conjugate flow (n' and q') are strictly equivalent to conditions along lines of curvature. (b) Conditions introducing rectangular components (n_0' and q_0') are always equivalent to conditions along lines of curvatures in Case IAa (normal flow given at both ends).

$C(\lambda_1)$ cannot be calculated directly, and a general condition for shear must always be

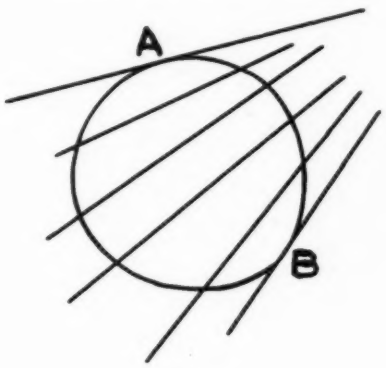


FIG. 6.

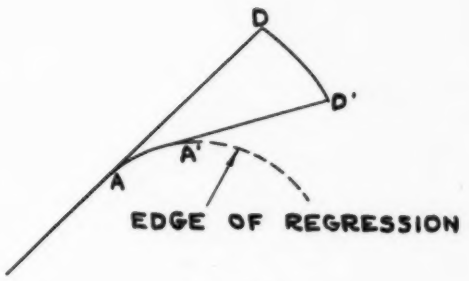


FIG. 7.

introduced. Moreover, for closed surfaces, it is necessary to have a univalued function $C(\lambda_1)$. It is to be noted that when there are no distributed loads, then

$$n_1 = 0, \quad n' = n_2 \sin \omega + 2n_{12} \cos \omega, \quad q' = n_{12}. \quad (17)$$

The generalized shear flow is then equal to the shear flow along the generators. This greatly simplifies the interpretation of the results. It is also to be noted that

$$\frac{n'}{\sin \omega} = n_2 + 2n_{12} \cot \omega. \quad (18)$$

This means that the force n_2 to be transmitted between two generators is *not equal* to the applied external load corresponding to the distance between the same generators.

(ID). The membrane is limited by a closed curve. This curve has two points, at least, where it is tangent to two generators at the surface (A and B, Fig. 6). In such points, the shear may be infinite and it is necessary to check for each condition of loading if that is the case.

Case II. The membrane includes its edge of regression (Fig. 7). As it has been seen, $C(\lambda_1)$ must be equal to zero in order to have a finite shear. If along the edge of regression, normal and shear flows are zero, and if on the other edges (generators or other curves) the shear and normal flow are not given, the membrane will be in equilibrium and $C = D = 0$ will determine the flow in any point. If the shear flow is not determined along the edge of regression, D is not determined, and C is equal to zero. Thus, in order to determine D, it is necessary to have conditions of the type: determined normal flow along the edge DD', undetermined shear and normal flows along AD and $A'D'$ (generators). Then the membrane is in equilibrium.

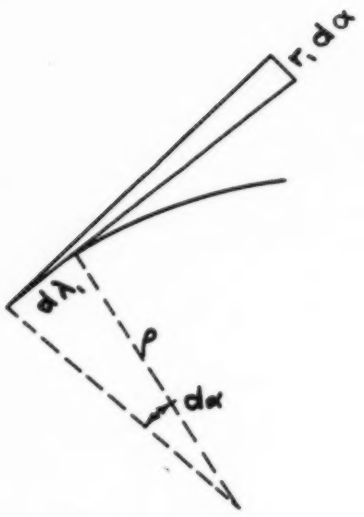


FIG. 8.

IV. DISTRIBUTION OF STRESS ALONG GENERATORS (NO DISTRIBUTED LOAD)

Equations (10) become in that case:

$$n_1=0,\quad n_{12}=\frac{C(\lambda_1)}{(\lambda_2-\lambda_1)^2},$$
$$n_2=\frac{\rho C'(\lambda_1)}{(\lambda_2-\lambda_1)^2}+\frac{\rho C(\lambda_1)}{(\lambda_2-\lambda_1)^3}+\frac{D(\lambda_1)}{\lambda_2-\lambda_1}. \tag{19}$$

λ_1 is the arc along the edge of regression, while $\lambda_2-\lambda_1$ is the radius of geodesic curvature r_1 of the curves λ_1 (see Fig. 1). Thus it can be written:

$$n_{12}=\frac{C(\lambda_1)}{r_1^2};\quad n_2=\frac{C'(\lambda_1)}{r_1^2}+\frac{\rho C(\lambda_1)}{r_1^3}+\frac{D(\lambda_1)}{r_1}. \tag{20}$$

It appears that n_{12} is given by the relation of shear in tapered panels, the apex of the panel being at the edge of regression. In order to examine n_2, consider, instead of the flow, the total force acting between two generators separated by an angle $d\alpha$. $r_1 d\alpha$ is the distance between the two generators at any point. Thus, the force is (noting that $\rho d\alpha=d\lambda_1$) (see Fig. 8)

$$dF_2=n_2 r_1 d\alpha=D(\lambda_1)d\alpha+\frac{C'(\lambda_1)d\lambda_1}{r_1}+\frac{C(\lambda_1)d\lambda_1}{r_1^2},$$
$$dF_2=D(\lambda_1)d\alpha+\frac{dC(\lambda_1)}{r_1}+\frac{C(\lambda_1)d\lambda_1}{r_1^2}. \tag{21}$$

The factor $D(\lambda_1)\cdot d\alpha$ corresponds to a direct load applied along the membrane between two generators. The factor $dC(\lambda_1):r_1$ is due to the variation of shear from one side of the considered element of membrane to the other. The factor $C(\lambda_1)d\lambda_1:r_1^2$ is due to the variation of position of the apex along the generators (as compared with a cone, in which case the apex has a definite position). Assuming that the generators are limited by the radii r_1', r_1'', with no end loads, then:

$$0=D(\lambda_1)d\alpha+\frac{dC(\lambda_1)}{r_1'}+\frac{C(\lambda_1)d\lambda_1}{r_1'^2};$$
$$0=D(\lambda_1)d\alpha+\frac{dC(\lambda_1)}{r_1''}+\frac{C(\lambda_1)d\lambda_1}{r_1''^2}.$$

F_{12}, total shear force which is transmitted along the generator, is given by

$$F_{12}=\int_{r_1'}^{r_1''}n_{12}dr_1=C(\lambda_1)\left(\frac{1}{r_1'}-\frac{1}{r_1''}\right);$$
$$C(\lambda_1)=F_{12}\frac{r_1'r_1''}{r_1''-r_1'}. \tag{22}$$

Thus, eliminating $D(\lambda_1)d\alpha$ and $C(\lambda_1)$:

$$\frac{dC(\lambda_1)}{d\lambda_1}\left(\frac{1}{r_1'}-\frac{1}{r_1''}\right)+F_{12}\left(\frac{1}{r_1'}+\frac{1}{r_1''}\right)=0;$$
$$\frac{dC(\lambda_1)}{d\lambda_1}=-F_{12}\frac{r_1''+r_1'}{r_1''-r_1'}. \tag{23}$$

This relation gives the variation of the distribution function $C(\lambda_1)$ with the position of the apex.

The value of $D(\lambda_1)$ is

$$D(\lambda_1)=\frac{\rho F_{12}}{(r_1''-r_1')}. \tag{24}$$

Thus

$$\frac{dF_2}{d\lambda_1}=\frac{F_{12}}{r_1''-r_1'}\left[1-\frac{r_1''+r_1'}{r_1}+\frac{r_1''r_1'}{r_1^2}\right]. \tag{25}$$

The maximum of this quantity is obtained for

$$\frac{2}{r_1}=\frac{1}{r_1'}+\frac{1}{r_1''} \tag{26}$$

and has the value

$$\left(\frac{dF_2}{d\lambda_1}\right)_m=\frac{F_{12}(r_1'-r_1'')}{4r_1'r_1''}=\frac{F_{12}}{4}\left(\frac{1}{r_1''}-\frac{1}{r_1'}\right). \tag{27}$$

Calling

$$\frac{r_1''}{r_1'}=t=\text{taper ratio};$$
$$L=r_1''-r_1'=\text{length of generator}, \tag{28}$$

it is found:

$$r_1'=\frac{L}{t-1};\quad r_1''=\frac{Lt}{t-1};\quad \frac{1}{r_1'}-\frac{1}{r_1''}=\frac{(t-1)^2}{Lt}. \tag{29}$$

and

$$d\lambda_1=-dr_1'=\frac{Ldt}{(t-1)^2}. \tag{30}$$

Thus

$$(dF_2)_m=-F_{12}\frac{dt}{4t}. \tag{31}$$

If the variation in taper is finite, it may be found, t' and t'' being the two tapers,

$$(\Delta F_2)_m=-F_{12}\frac{\sqrt{t''}-\sqrt{t'}}{\sqrt{t''}+\sqrt{t'}}, \tag{32}$$

which, at the limit, gives the relation (31).

If instead of considering F_2, n_2 is introduced, it is found

$$\frac{n_2 d\alpha}{d\lambda_1}=\frac{F_{12}}{r_1''-r_1'}\left[\frac{1}{r_1}-\frac{r_1''+r_1'}{r_1^2}+\frac{r_1''r_1'}{r_1^3}\right], \tag{33}$$

which is maximum for

$$\frac{3}{r_1}=\frac{1}{r_1'}+\frac{1}{r_1''}+\left(\frac{1}{r_1'^2}+\frac{1}{r_1''^2}-\frac{1}{r_1'r_1''}\right)^{\frac{1}{2}} \tag{34}$$

and has the value

$$\begin{aligned}(n_2 d\alpha)_m=&-\left[-\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)+\frac{2}{3}\left(t^{\frac{1}{2}}+\frac{1}{t^{\frac{1}{2}}}\right)\right.\\&\left.+\frac{2}{3}\left(t+\frac{1}{t}-1\right)^{\frac{1}{2}}\right]\frac{1}{9}\frac{F_{12}}{L}\frac{dt}{t}\frac{1}{\sqrt{t}-\frac{1}{\sqrt{t}}},\\(n_2)_m=&\frac{-\rho}{9L^2}F_{12}\left[-\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)+\frac{2}{3}\left(t^{\frac{1}{2}}+\frac{1}{t^{\frac{1}{2}}}\right)\right.\\&\left.+\frac{2}{3}\left(t+\frac{1}{t}-1\right)^{\frac{1}{2}}\right]\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right).\end{aligned} \tag{35}$$

n_2 and F_2 are compressions when the shear is directed as indicated on Fig. 9.

When the normal flows n_2 at the ends of the generator are different from zero [this generally happens when the edges do not coincide with the lines of curvature, see Eqs. (17), (18)], it is necessary to determine the points of zero normal flow (which can be out of the portion of the membrane limited by the edges), and apply the above formulas between these points.

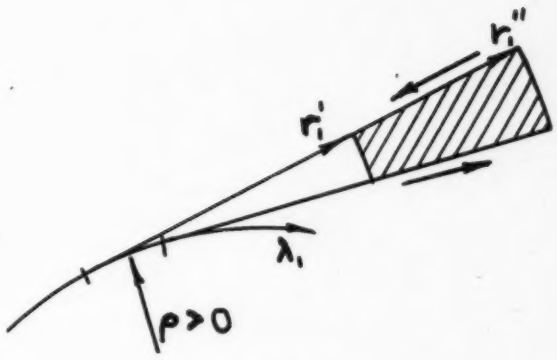


FIG. 9.

Thus, being given at both ends of the generator (see Fig. 10),

$$\begin{aligned}n_2'&=\frac{n'-2n_{12}'\cos\omega'}{\sin\omega},\\n_2''&=\frac{n''-2n_{12}''\cos\omega''}{\sin\omega''},\end{aligned} \tag{36}$$

the equations giving C and D become

$$\begin{aligned}r'd\alpha n_2'&=Dd\alpha+\frac{dC(\lambda_1)}{r_1'}+\frac{C(\lambda_1)d\lambda_1}{r_1'^2},\\r''d\alpha n_2''&=Dd\alpha+\frac{dc(\lambda_1)}{r_1''}+\frac{c(\lambda_1)d\lambda_1}{r_1''^2}.\end{aligned}$$

Equation (22) remains unchanged while (23) becomes

$$\frac{dC(\lambda_1)}{d\lambda_1}\left(\frac{1}{r_1'}-\frac{1}{r_1''}\right)+F_{12}\left(\frac{1}{r_1'}+\frac{1}{r_1''}\right)=\frac{(n_2'r'-n_2''r'')}{\rho} \tag{37}$$

and

$$D(\lambda_1)=\frac{n_2''r_1''^2-n_2'r_1'^2}{r_1''-r_1'}+\frac{\rho F_{12}}{r_1''-r_1'}. \tag{38}$$

Thus

$$\begin{aligned}\frac{dF_2}{d\lambda_1}=&\frac{F_{12}}{r_1''-r_1'}\left[1-\frac{r_1''+r_1'}{r_1}+\frac{r_1''r_1'}{r_1^2}\right]+\frac{1}{\rho(r_1''-r_1')}\\&\times\left[n_2''r_1''^2\left(1-\frac{r_1'}{r_1}\right)-n_2'r_1'^2\left(1-\frac{r_1''}{r_1}\right)\right],\end{aligned} \tag{39}$$

$$\begin{aligned}\left(\frac{1}{r_1'}-\frac{1}{r_1''}\right)\frac{\rho dF_2}{d\lambda_1}=&\rho F_{12}\left(\frac{1}{r_1'}-\frac{1}{r_1}\right)\left(\frac{1}{r_1''}-\frac{1}{r_1}\right)\\&+n_2''r_1''\left(\frac{1}{r_1'}-\frac{1}{r_1}\right)-n_2'r_1'\left(\frac{1}{r_1''}-\frac{1}{r_1}\right).\end{aligned} \tag{40}$$

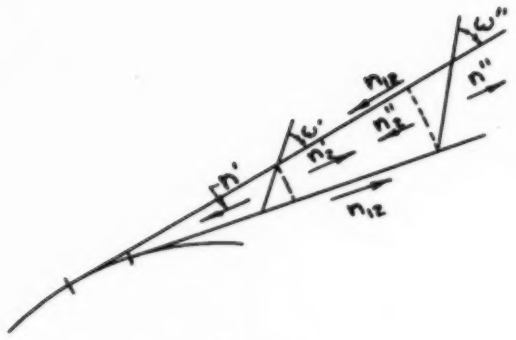


FIG. 10.

In order to determine the points of zero value of $dF_2/d\lambda_1$, the following unknown is taken:

$$2/r_1-1/r_1'-1/r_1''=x. \tag{41}$$

This leads to the equation

$$x^2\rho F_{12}-2x(n_2''r_1''-n_2'r_1')-\rho F_{12}\left(\frac{1}{r_1'}-\frac{1}{r_1''}\right)^2 +2(n_2'r_1'+n_2''r_1'')\left(\frac{1}{r_1'}-\frac{1}{r_1''}\right)=0. \tag{42}$$

The values x' and x'' thus found give the locations of the points of zero $dF_2/d\lambda_1$. The point of maximum $dF_2/d\lambda_1$ is given by x corresponding to the average of the two values x', x'', and corresponds to

$$x_0=\frac{n_2''r_1''-n_2'r_1'}{\rho F_{12}}. \tag{43}$$

Thus

$$\frac{2}{r_1}=\frac{1}{r_1'}+\frac{1}{r_1''}+\frac{n_2''r_1''-n_2'r_1'}{\rho F_{12}}. \tag{44}$$

For that point,

$$\left(\frac{dF_2}{d\lambda_1}\right)_m=\left[\left(\frac{1}{r_1''}-\frac{1}{r_1'}\right)\frac{F_{12}}{4}+\frac{n_2''r_1''+n_2'r_1'}{2\rho}\right] +\frac{(n_2''r_1''-n_2'r_1')^2}{4\rho^2F_{12}\left(\frac{1}{r_1''}-\frac{1}{r_1'}\right)}. \tag{45}$$

So far an isolated generator has been examined. The results can be extended to a complete surface under determined conditions of loading. Hereafter, the condition of shear transmission on a surface limited by two lines of curvature is examined. For that purpose, equations written as (19) are used. Repeating the same calculation as before, Eqs. (22), (23) are found and, eliminating F_{12} between those two equations and replacing r in function of λ, it is found:

$$\frac{1}{C(\lambda_1)}\frac{dC(\lambda_1)}{d\lambda_1}=-\frac{r_1''+r_1'}{r_1'r_1''}=-\left(\frac{1}{r_1'}+\frac{1}{r_1''}\right) =-\frac{1}{\lambda_2'-\lambda_1}+\frac{1}{\lambda_2''-\lambda_1}. \tag{46}$$

By integration,

$$\log C(\lambda_1)=\log(\lambda_2'-\lambda_1)(\lambda_2''-\lambda_1),$$

or

$$C(\lambda_1)=K(\lambda_2'-\lambda_1)(\lambda_2''-\lambda_1)=Kr_1'r_1'' =F_{12}r_1'r_1'':(r_1''-r_1'), \tag{47}$$

$$C'(\lambda_1)+K[(\lambda_2'-\lambda_1)+(\lambda_2''-\lambda_1)]=0. \tag{48}$$

Then

$$n_{12}=\frac{K(\lambda_2'-\lambda_1)(\lambda_2''-\lambda_1)}{(\lambda_2-\lambda_1)^2} =\frac{Kr_1'r_1''}{r_1^2}=\frac{F_{12}r_1'r_1''}{r_1^2(r_1''-r_1')}, \tag{49}$$

$$n_2=-\frac{\rho(\lambda_2''+\lambda_2'-2\lambda_1)K}{(\lambda_2-\lambda_1)^2} +\frac{\rho(\lambda_2'-\lambda_1)(\lambda_2''-\lambda_1)K}{(\lambda_2-\lambda_1)^3}+\frac{D(\lambda_1)}{\lambda_2-\lambda_1}$$

$$=\frac{\rho K}{(\lambda_2-\lambda_1)^2}[(\lambda_2'-\lambda_2)(\lambda_2''-\lambda_2) -(\lambda_2-\lambda_1)^2]+\frac{D(\lambda_1)}{\lambda_2-\lambda_1}.$$

In order to have $n_2=0$ at the ends, $D(\lambda_1)=\rho K$. Thus:

$$n_{12}=\frac{\rho K(r_1'-r_1)(r_1''-r_1)}{r_1^3} =\frac{\rho F_{12}(r_1'-r_1)(r_1''-r_1)}{(r_1''-r_1')r_1^3}. \tag{50}$$

The two formulae (49), (50) thus solve the problem of shear transmission. In practical applications, the problem of shear transmission is stated as follows. The normal loads applied at the end of the generators being given, determine the stress distribution at every point of the surface. Let $n_2'(\lambda_1)$, $n_2''(\lambda_1)$ be the values of n_2 at the end of the generators. Then we may write the last equation (19) applied at the two

ends of each generator as:

$$(\lambda_2'-\lambda_1)n_2'(\lambda_1)=\frac{\rho C'(\lambda_1)}{\lambda_2'-\lambda_1}+\frac{\rho C(\lambda_1)}{(\lambda_2'-\lambda_1)^2}+D(\lambda_1),$$
$$(\lambda_2''-\lambda_1)n_2''(\lambda_1)=\frac{\rho C'(\lambda_1)}{\lambda_2''-\lambda_1}+\frac{\rho C(\lambda_1)}{(\lambda_2''-\lambda_1)^2}+D(\lambda_1). \quad (51)$$

Taking the difference of the equations,

$$(\lambda_2'-\lambda_1)n_2'(\lambda_1)-(\lambda_2''-\lambda_1)n_2''(\lambda_1)$$
$$=\rho C'(\lambda_1)\left[\frac{1}{\lambda_2'-\lambda_1}-\frac{1}{\lambda_2''-\lambda_1}\right]$$
$$+C(\lambda_1)\left[\frac{1}{(\lambda_2'-\lambda_1)^2}-\frac{1}{(\lambda_2''-\lambda_1)^2}\right].$$

Writing $C(\lambda_1)$ in the form:

$$C(\lambda_1)=K(\lambda_1)(\lambda_2'-\lambda_1)(\lambda_2''-\lambda_1),$$

the above equation becomes with the new unknown $K(\lambda_1)$:

$$\frac{\rho dK(\lambda_1)}{d\lambda}(\lambda_2''-\lambda_2')=(\lambda_2'-\lambda_1)n_2'(\lambda_1)$$
$$-(\lambda_2''-\lambda_1)n_2''(\lambda_1). \quad (52)$$

By integration $K(\lambda_1)$ is determined, and replacing it in function of $C(\lambda_1)$ it is found:

$$C(\lambda_1)=\frac{(\lambda_2'-\lambda_1)(\lambda_2''-\lambda_1)}{\lambda_2''-\lambda_2'}\int[(\lambda_2'-\lambda_1)n_2'(\lambda_1)$$
$$-(\lambda_2''-\lambda_1)n_2''(\lambda_1)]\frac{d\lambda_1}{\rho}. \quad (53)$$

Thus

$$n_{12}=\frac{(\lambda_2'-\lambda_1)(\lambda''-\lambda_1)}{(\lambda_2''-\lambda_2')(\lambda_2-\lambda_1)^2}\int[(\lambda_2'-\lambda_1)n_2'(\lambda_1)$$
$$-(\lambda_2''-\lambda_1)n_2''(\lambda_1)]\frac{d\lambda_1}{\rho} \quad (54)$$

and by substitution in the expression of n_2 and determination of $D(\lambda_1)$ by one of the equations (52), n_2 is found.

$$n_2=\frac{(\lambda_2''-\lambda_2)(\lambda_2'-\lambda_1)^2}{(\lambda_2''-\lambda_2')(\lambda_2-\lambda_1)^2}n_2'(\lambda_1)$$
$$+\frac{(\lambda_2'-\lambda_2)(\lambda_2''-\lambda_1)^2}{(\lambda_2'-\lambda_2'')(\lambda_2-\lambda_1)^2}n_2''(\lambda_1)$$
$$+\frac{(\lambda_2'-\lambda_1)(\lambda_2''-\lambda_1)\rho}{(\lambda_2-\lambda_1)^2(\lambda_2''-\lambda_2')}\int[(\lambda_2'-\lambda_1)n_2'(\lambda_1)$$
$$-(\lambda_2''-\lambda_1)n_2''(\lambda_1)]\frac{d\lambda_1}{\rho}. \quad (55)$$

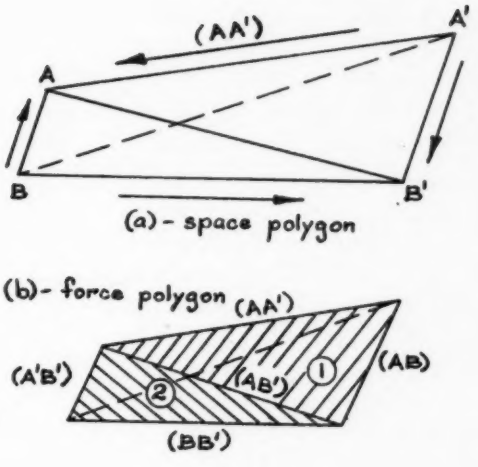


FIG. 11.

V. PRACTICAL APPLICATIONS TO AIRPLANE STRUCTURES

In actual structures, such as wings or fuselages of airplanes, the developable membrane which is constituted by a skin-stressed structure is divided into segments by redistribution frames. Each segment between frames can be analyzed by means of the methods developed thus far, provided the edge loads are given in such a way as to render the segment statically determined.

From the examination of the edge conditions, it appears that it is necessary to determine axial load distribution in the direction of the generators along both frames limiting a segment. Thus, the knowledge of axial distributions at each frame renders the structure statically determined.

Since the distribution of axial loads is not known except at the end frames, it is necessary to introduce a method of distribution of those loads. For structures of elongated shape in the general direction of the generators, having parallel frames perpendicular to such general direction, hereafter called axis of the structure, the beam distribution theory is most convenient, since it defines axial loads in a section by the properties of the structure at that section.

It is to be borne in mind that such method of distribution is arbitrary, at least for two reasons: (a) Beam distribution applies only for cylindrical beams, and theoretically, for constant shear loads only; (b) from the distribution of axial loads along a generator which has been established in the preceding paragraph, it appears that if a

beam distribution exists at two end frames, it does not subsist at intermediate sections, which shows the arbitrary character of the assumption of beam distribution at redistribution frames only. The beam distribution at frames is justified by its simplicity and by the existence of phenomena of buckling and plasticity, which prevent the determination of effective areas with a degree of certainty which would warrant a research of actual distributions.

Omitting the conditions which render a beam distribution acceptable in the case of a cylinder, it is always possible to check the legitimacy of the extension to developable surfaces by calculating the axial load distributions at various sections between frames. If such distributions are sufficiently close to beam distributions, the method can be considered as acceptable within the degree of accuracy required. If not, the method must be reduced to the function of determining stresses inside a structure—especially shears—which are compatible with applied edge loads.

The beam distribution method defines components of the loads along the axis of the structure; if α is the angle between a generator and the axis the quantity thus determined is $\bar{n} = n \cos \alpha$, n being defined on Fig. 5. Substituting in Eq. (18), it is found:

$$n_2 = \frac{\bar{n}}{\cos \alpha \sin \omega} - 2n_{12} \cot \omega. \qquad (56)$$

The actual axial flow n_2 being different from the flow $\bar{n}$ defined by the beam theory, there is an additional reason of discrepancy between the actual distribution and the distribution determined by the proposed method. The discrepancy can be reduced by multiplying actual areas by the factor $\cos^3 \alpha \sin^2 \omega$ which compensates, in a calculation by a method of least work, for the ratio between n_2 and $\bar{n}$ and for the actual length of generators, but no correction is possible for the term containing n_{12} in Eq. (56).

VI. GRAPHICAL METHODS OF ANALYSIS

In the following paragraphs, it is assumed that surface loads are negligible. In actual problems of stressed-skin structures, the continuous distribution of material is, for convenience, replaced by concentrated areas; the method

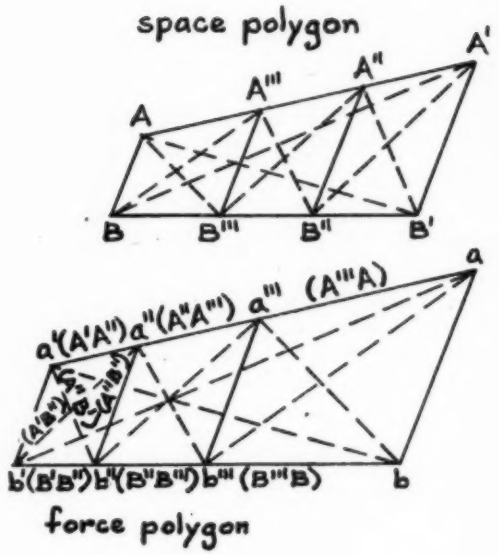


FIG. 12.

is especially accurate when the structure is composed of longitudinal stringers (members) and a skin, an effective area of which is supposed to be concentrated at the members, the skin between members transmitting shear only. Then the developable panel between two members is replaced by a plane panel; thus, the given membrane is replaced by a polyhedron composed of small tapered panels. The error resulting from that substitution consists of the non-coincidence between the line of action of shear and the chord of the panel; the difference is generally negligible but can be taken into consideration.

The equilibrium of a tapered panel belonging to a polyhedron is defined as soon as the shearing force along one of its edges is given, when it is assumed that no axial force can act along the edges. This results from the fact that along the edges in the direction of AA', BB', there can be no load because of the angle between the panels (equivalent to $n_1 = 0$ in the developable membrane), and from the distribution of axial loads which are considered as concentrated loads along the lines AA', BB' (Fig. 11). Since all loads on the panel are pure shearing forces, there are 3 unknowns (shearing forces on 3 sides), and the equilibrium can be obtained by conditions of statics. Graphical analysis can be applied to that problem in a very simple manner. Replacing the panel by a latticed structure by adding one

of the two diagonals AB' or $A'B$, the equilibrium is obtained as follows:

In A, forces (AB), (AA'), (AB') must be in equilibrium, which defines the values of (AB') and (AB) [see triangle (1)]. In B', forces (AB'), $(A'B')$, (BB') must be in equilibrium, which defines the values of $(A'B')$ and (BB') [see triangle (2)]. For checking purpose, the diagonal $A'B$ can be used, which defines the two other triangles of the force polygon. The space polygon and the force polygon are similar figures but it must be noted that: *The side AB of the space polygon is transmitting a shearing force (AB) which in the force polygon corresponds to $A'B'$: To the small side corresponds the large force, and reciprocally.* That property is the result of the well-known relation in tapered panels: The shear flow is inversely proportional to the square of the length of the side. Since the shearing force is equal to the product of the shear flow by the length of the side, it appears that the force is inversely proportional to the length of the side. It is to be noted that this is valid for the two parallel sides only, and on the non-parallel sides, shearing forces are proportional to the sides, while shear stresses are constant. Depending on the conventions of sign for shear and the sense of the end forces which produce the shear, the force polygon can be placed in two opposite manners, which result one from the other by a rotation of 180°.

When the shearing force transmitted between two points along a generator must be determined, the graphical construction is the following (Fig. 12).

The equilibrium of the panel being determined, the load along $A'A''$, for instance, is determined as follows: The line $A''B''$, parallel to AB, $A'B'$ is drawn and the equilibrium of the panel $A'B'A''B''$ is determined in similar manner to the equilibrium of the whole panel. The only difference is that the starting line is $(A'B')$. The two diagonals $A''B'$ and $A'B''$ are drawn and the force along $A''B''$ is determined (passes by $a''b''$).

If the load between two interior points of a generator $A''A'''$, for instance, must be determined, the construction is repeated between those points, starting from $A''B''$, in the same manner as previously made for the panel $A'B'A''B''$. A final check can be made by determining the load on AB, starting from $A'''B'''$, or reciprocally. Figure 12 is self-explanatory for the sequence of operations made.

The analysis of a panel is thus defined when the shearing force transmitted from an edge member is known. That shearing force is the sum of the shearing force transmitted by the adjacent panel and the difference of the end loads on the member between the two panels. It is thus possible to proceed panel by panel, provided there is an edge panel, i.e., provided the membrane is open. If the membrane is closed, the same problem of the determination of the origin of shear exists as in closed cylinders.

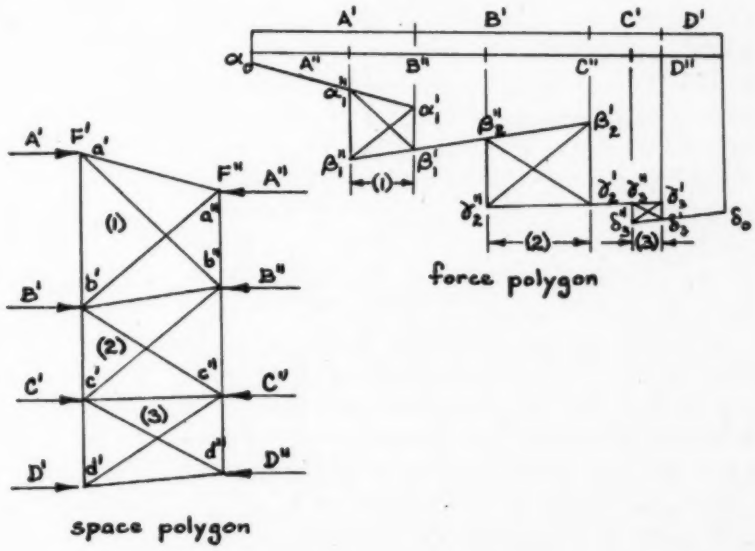


FIG. 13.

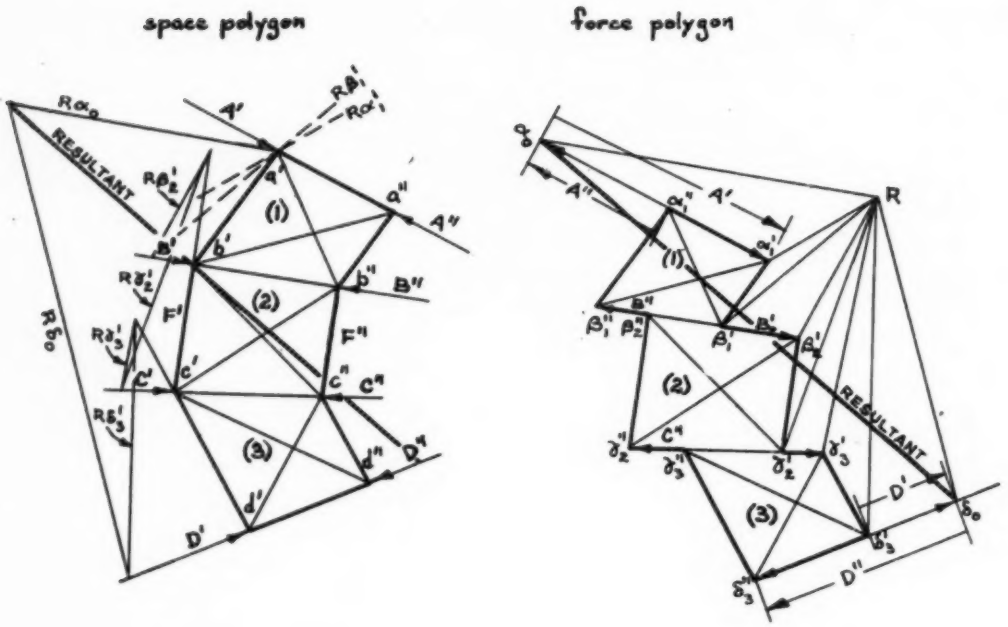


FIG. 14.

The analysis can be made on the membrane developed on a plane. However, the use of a plane of projection generally simplifies the problem and gives more information on the equilibrium. Two types of projection can be used.

(a) Projection on a Plane Perpendicular to the Frames

The frames appear as parallel lines and the axial loads acting on the members are determined by a beam distribution if they are not known in terms of applied forces on an end frame. It is to be noted that if the beam distribution gives loads parallel to the axis, the loads thus defined can be directly used. The method is the following (Fig. 13). The axial components of the loads in the members are placed end to end on a line parallel to the axis, the forces at one frame being placed first, the forces at the other frame being placed at the end of the forces at the first frame, as shown (A', B', C', D' = forces at frame F'; A'', B'', C'', D'' = forces at frame F'').

On panel (1) the acting load is $\mathbf{A'}+\mathbf{A''}$, thus the force equilibrium of (1) is drawn between the ends of A' and A'', as shown. Drawing an arbitrary line ($\alpha_1''\alpha_1'$) parallel to $a'a''$, the equilibrium of panel (1) is defined as indicated previously by drawing through α_1', α_1'' parallels to the sides and diagonals corresponding to a'', a' of panel (1) and closing with a parallel ($\beta_1'\beta_1''$) to the side $b'b''$. The inversion of correspondence between a', α_1'' and a_1'', α_1' must be noted. Thus the load along $b'b''$, in panel (1), is $\beta_1''\beta_1'$. Adding forces $\mathbf{B'}$ and $\mathbf{B''}$, it appears that the load along $b'b''$, in panel (2), is $\beta_2''\beta_2'$, on the same line as $\beta_1''\beta_1'$, but determined by the intersection with the lines parallel to the frames drawn through the ends of $\mathbf{B'}$, $\mathbf{B''}$.

From $\beta_2''\beta_2'$, panel (2) can be analyzed in same manner as panel (1) and procedure can be continued for each panel.

In Fig. 13, the projections of the loads in members and of the shearing forces are, on frame F', for instance,

member	$a'a''$	$b'b''$	$c'c''$	$d'd''$
load	$\alpha_0\alpha_1'$	$\beta_1'\beta_2'$	$\gamma_2'\gamma_3'$	$\delta_3'\delta_0$
side	$a'b'$	$b'c'$	$c'd'$	
shearing force .	$\alpha_1'\beta_1'$	$\beta_2'\gamma_2'$	$\gamma_3'\delta_3'$	

Dividing the shearing forces by the lengths of the sides, the shear flow along the frame is determined.

This method gives shear flows, but does not define the locations of resultants. It has the disadvantage of being inaccurate for panels nearly perpendicular to the plane of projection. Its advantage resides in the fact that it is quicker than the following method, and it is well adapted for symmetric structures under symmetric loads, using the plane of symmetry

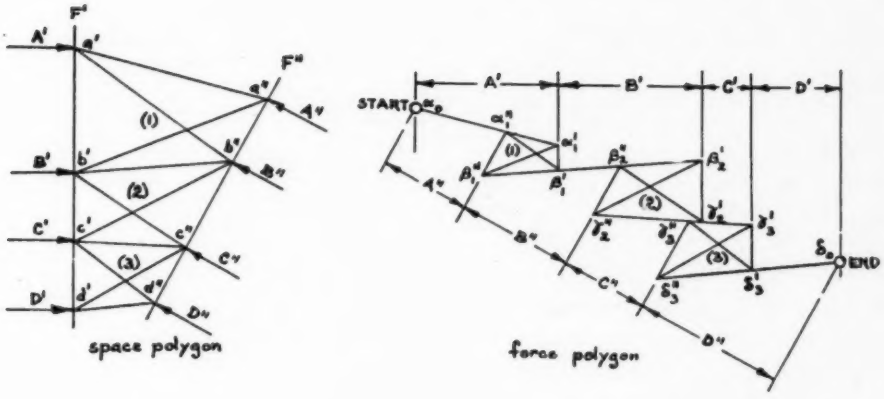


FIG. 15.

as plane of projection. It is to be noted that when a pure shear equilibrium is examined, the force polygon becomes a figure similar to the space polygon.

(b) Projection on a Plane Parallel to the Frames

In this case components of the end loads parallel to the frames are required. Δ_n being the projected length of a member n on the frames, L the distance between frames, F_n', F_n'' the axial loads in the member n at the end frames, the projections on the planes of the frames are:

$$f_n' = F_n'\Delta_n : L; \quad f_n'' = F_n''\Delta_n : L. \qquad (57)$$

Knowing the loads f_n', f_n'', force diagrams can be determined by a method similar to that of Fig. 13.

Figure 14 shows the procedure. Loads at the ends of $a'a''$, called **A'**, **A''**, are drawn end to end. Their sum is $\alpha_1''\alpha_1'$. The equilibrium of panel (1) is determined by drawing through α_1'' and α_1' parallels to the sides and diagonals corresponding to a' and a''. The operations from there on are similar to those of Fig. 13, but the loads are not represented along a base line, and must be measured along each member. The directions of the loads must be chosen in such a manner that in the completed figure the sequence be, for instance, **D''**, **C''**, **B''**, **A''**, **A'**, **B'**, **C'**, **D'**, which shows that two opposite forces B', B'' are measured in same direction along their line of support $\beta_1''\beta_1'$. It results that compression forces are measured in one direction, while tension forces are measured in opposite direction. Thus, in Fig. 14, starting from the top, compression forces are drawn from F' toward F'' and tension forces are drawn from F'' toward F'. Once the force polygon is completed, the value of the resultant of the shears acting along the frames is determined. It is the force $\alpha_0\delta_0$. Its location is determined by a classical construction of addition of forces, which is represented in Fig. 14, using an arbitrary pole R and the forces acting on the frame F'. The resultant is shown in position. A check can be made by repeating the same operation at the other frame.

By combining various unit solutions, shear centers or stress centers can be determined by intersection of two such resultants. Then, arms can be measured for the calculation of moment coefficients for torsion bending and the torsion bending constant may be determined. In case of closed profile, a cut is always made in order to start the calculation, and a unit solution is made under pure shear transmission. If in Fig. 14 the force placed on $a'a''$ is equal to $a'a''$ at a certain scale, the space polygon for shear transmission becomes the force polygon. From that force polygon and an arbitrary pole the resulting couple can be determined.

(c) Extension to Non-Parallel Frames

The equilibrium of a developable membrane can be examined between two non-parallel

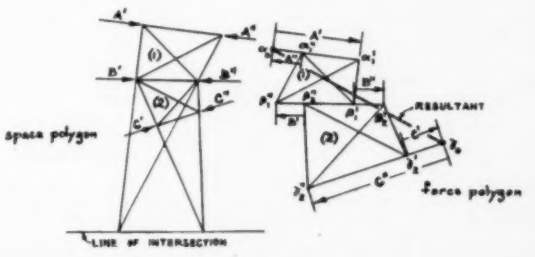
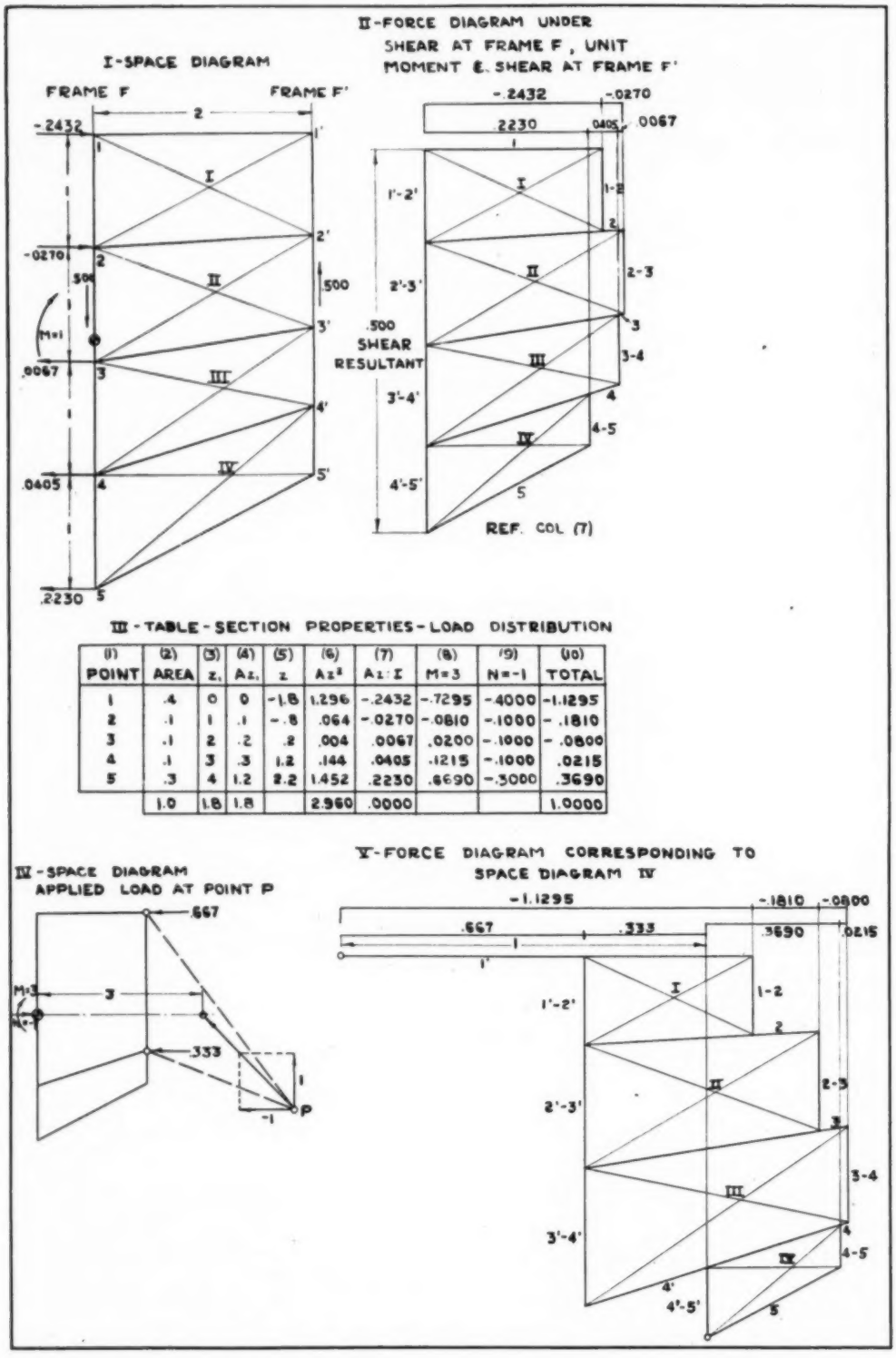


FIG. 16.


I-SPACE DIAGRAM
FRAME F
FRAME F'
II-FORCE DIAGRAM UNDER SHEAR AT FRAME F, UNIT MOMENT & SHEAR AT FRAME F'
SHEAR RESULTANT
REF. COL (7)
III-TABLE-SECTION PROPERTIES-LOAD DISTRIBUTION
IV-SPACE DIAGRAM APPLIED LOAD AT POINT P
V-FORCE DIAGRAM CORRESPONDING TO SPACE DIAGRAM IV

(1) POINT	(2) AREA	(3) z_1	(4) Az_1	(5) z	(6) Az^2	(7) $Az \cdot I$	(8) M=3	(9) N=-1	(10) TOTAL
1	.4	0	0	-1.8	1.296	-.2432	-.7295	-.4000	-1.1295
2	.1	1	.1	-.8	.064	-.0270	-.0810	-.1000	-.1810
3	.1	2	.2	.2	.004	.0067	.0200	-.1000	-.0800
4	.1	3	.3	1.2	.144	.0405	.1215	-.1000	.0215
5	.3	4	1.2	2.2	1.452	.2230	.6690	-.3000	.3690
	1.0	1.8	1.8		2.960	.0000			1.0000

FIG. 17. Equilibrium of a developable membrane in projection on a plane perpendicular to the frames.

frames. Special precautions must be taken for the distribution of the axial forces along a frame which is not parallel to the axis of the surface, and the method is simple only in the case when such a frame is an end frame, on which applied forces are known.

(1) *Projection on a Plane Perpendicular to the Frames*

The procedure is essentially the same as between two parallel frames, the difference between Fig. 15 and Fig. 13 residing in the fact that the two load lines giving the values of the loads perpendicular to the frames are not parallel. Thus, it is necessary to draw two directions of sets of lines defining the loads at the two frames, respectively parallel to the two frames. The force polygon is drawn starting at the intersection of the two first lines, defining the origins of loads A' and A'', respectively, and it must end at the intersection of the two last lines, which gives a control of the equilibrium. The detail of the force polygon is the same as in Fig. 13, and forces and shears are measured in the same manner. It is to be noted that force and space panels are not similar figures in this case, although all their sides and diagonals are parallel two by two.

(2) *Projection on a Plane Parallel to the Intersection of the Two Frames*

The construction Fig. 16 is comparable to that of Fig. 14 for parallel frames. It differs by the fact that the intersections of each panel with both frames, instead of being parallel, intersect along the line of intersection of the two frames. The construction of the force polygon is identical in both cases. As in Fig. 15, space and force polygon are not similar. For open structures, resultants and points of application can be determined in a manner similar to that corresponding to parallel frames. For closed structures an important difference must be emphasized: Distribution corresponding to shear without end axial loads does not correspond to a torque, but to a *resultant* which is located along the line of intersection of the frames. This appears from the fact that since end loads are shears, their respective resultants must lie in the planes of the end frames. As they are in equilibrium, they must necessarily be located along the line of intersection of their planes.

(d) Examples of Application

The basic method represented in Fig. 13 is applied in Fig. 17 to a structure composed of 4 panels and 5 members. The method is worked out in two conditions. (1) A shearing force applied at an end frame, in equilibrium with the opposite force and a couple distributed according to the beam theory at the other frame; (2) a load applied by means of two members to an end of the structure, in equilibrium with reactions distributed according to the beam theory (axial resultant and resulting couple) at the other frame.

Calendar of Meetings

July
16–21 American Society of Civil Engineers, Chicago, Illinois
22 American Physical Society, Berkeley, California

August
29–Sept. 1 American Institute of Electrical Engineers, Los Angeles, California

September
11–16 American Association for the Advancement of Science, Cleveland, Ohio
13–15 Society of Automotive Engineers, Milwaukee, Wisconsin

October
5– 6 American Institute of Mining and Metallurgical Engineers, Pittsburgh, Pennsylvania
5– 7 National Electronics Conference, Chicago, Illinois
5– 7 Society of Automotive Engineers, Los Angeles, California
6– 9 Electronic Parts and Equipment Industry Conference, Chicago, Illinois
9–14 American Society of Civil Engineers, Cleveland, Ohio
16–18 American Institute of Mining and Metallurgical Engineers, Cleveland, Ohio
16–20 American Society for Metals, Cleveland, Ohio
16–21 American Welding Society, Cleveland, Ohio
20–21 Optical Society of America, New York, New York
28 American Mathematical Society, New York, New York

November
2– 3 Society of Automotive Engineers, Tulsa, Oklahoma
17–18 Society of Rheology, New York, New York

General Solution of Two-Dimensional Problems of Elasticity

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(Received April 22, 1944)

THE solution of the general problem of elasticity given hereafter applies for a two-dimensional simply connected domain with an arbitrary boundary and given volume and boundary forces.

Since the general problem for an indefinite plane with various nuclei of strain has been solved,[1,2] it is possible to find a solution which would satisfy the differential equations of equilibrium inside the domain without satisfying the conditions at the boundary. Then, by superposition, it is possible to reduce the problem to that of volume forces equal to zero and given boundary forces. Let

$\Sigma(\sigma_x, \sigma_y, \tau_{xy})$ = the stresses in rectangular coordinates (x, y) which satisfy both volume and boundary conditions.

$\Sigma^0(\overset{0}{\sigma}_x, \overset{0}{\sigma}_y, \overset{0}{\tau}_{xy})$ = the stresses satisfying volume conditions only.

X, Y the components of the boundary forces per unit length $\bar{\Sigma}(\bar{\sigma}_x, \bar{\sigma}_y, \bar{\tau}_{xy})$ = stresses defined by the relation:

$$\bar{\Sigma} = \Sigma - \Sigma^0. \tag{1}$$

Those stresses $\bar{\Sigma}$ satisfy the conditions which result by taking the difference of the equations of equilibrium satisfied by the systems Σ and Σ^0 (see Fig. 1).

Since Σ and Σ^0 both satisfy the volume conditions, stresses $\bar{\Sigma}$ are in equilibrium with zero volume forces.

Boundary forces $\bar{X}$, $\bar{Y}$ which are in equilibrium with $\bar{\Sigma}$ are given by the relations:

$$\begin{aligned} \bar{X} &= X - \overset{0}{\sigma}_x \cos(n, x) - \overset{0}{\tau}_{xy} \cos(n, y), \\ \bar{Y} &= Y - \overset{0}{\tau}_{xy} \cos(n, x) - \overset{0}{\sigma}_y \cos(n, y), \end{aligned} \tag{2}$$

in which n denotes the outer normal to the boundary.

Thus, the general solution of two-dimensional problems of elasticity is determined by superposition of two solutions: One (Σ^0) is that of the equilibrium of an indefinite domain with given volume forces. The other ($\bar{\Sigma}$) is that of the equilibrium of a bounded domain with given boundary forces and zero volume forces. This is the problem which is examined hereafter.

It is known that the stresses $\bar{\Sigma}$ can be expressed by means of an Airy stress function Φ:[3,4]

$$\bar{\sigma}_x = \frac{\partial^2\Phi}{\partial y^2}; \quad \bar{\sigma}_y = \frac{\partial^2\Phi}{\partial x^2}; \quad \bar{\tau}_{xy} = -\frac{\partial^2\Phi}{\partial x \partial y} \tag{3}$$

and that the stress function is a biharmonic function:

$$\nabla^2\nabla^2\Phi = 0. \tag{4}$$

At the boundary, the stress function is completely defined by Eqs. (2); calling s the arc along the boundary, Eqs. (2) become:[4]

$$\begin{aligned} \frac{\partial^2\Phi}{\partial y^2}\frac{dy}{ds} + \frac{\partial^2\Phi}{\partial x \partial y}\frac{dx}{ds} &= \bar{X}, \\ \frac{\partial^2\Phi}{\partial x^2}\frac{dx}{ds} + \frac{\partial^2\Phi}{\partial x \partial y}\frac{dy}{ds} &= -\bar{Y}; \end{aligned} \tag{5}$$

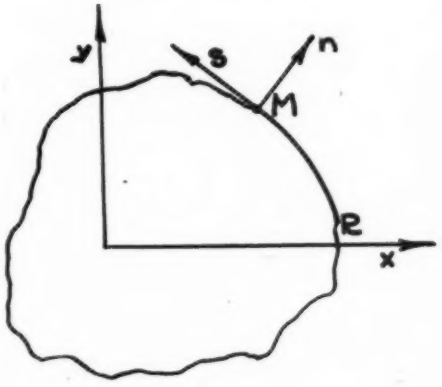

FIG. 1.

[1] S. Timoshenko, *Theory of Elasticity* (1934), p. 108.
[2] A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity* (1927), fourth edition, p. 208.
[3] Reference 1, p. 25.
[4] E. Trefftz, "Mathematische Elastizitaetstheorie," in *Handbuch der Physik* Volume VI; *Mechanik der Elastischen Koerper* (1928), p. 109.

by integration, introducing the notations $P(s)$, $Q(s)$:

$$\frac{\partial\Phi}{\partial y}=\int Xds=P(s);\quad \frac{\partial\Phi}{\partial x}=-\int Yds=-Q(s);\quad (6)$$

$$\Phi=\int\left(\frac{\partial\Phi}{dy}\frac{dy}{ds}+\frac{\partial\Phi}{\partial x}\frac{dx}{ds}\right)ds$$

$$=\int(Pdy-Qdx)=f(s).\quad (7)$$

The derivative of Φ along the outer normal is determined by combination:

$$\frac{d\Phi}{dn}=\frac{\partial\Phi}{\partial x}\frac{dx}{dn}+\frac{\partial\Phi}{\partial y}\frac{dy}{dn}=-\left(Q\frac{dy}{ds}+P\frac{dx}{ds}\right)=g(s).\quad (8)$$

It can be shown that $\Phi(M)$ at any point M of the boundary represents the moment in respect to M of all the boundary forces acting between a fixed point R and M, while $d\Phi(M):dn$ represents the value of the tangential component, at M, of the same boundary forces.

Thus, the problem of two-dimensional elasticity is reduced to the research of a biharmonic function Φ satisfying the boundary conditions:

$$\Phi(s)=f(s);\quad \frac{d\Phi(s)}{dn}=g(s).\quad (9)$$

The function Φ can be considered as a potential of volume distribution of masses with a harmonic continuous density $\mu/2\pi$:[5]

$$\nabla^2\Phi=\mu\quad (10)$$

$$\nabla^2\mu=0.\quad (11)$$

Then the general solution for Φ can be written as the sum of a solution of (10) (μ being a general harmonic function) and a general harmonic function Φ_1:

$$\nabla^2\Phi_1=0.\quad (12)$$

The general solution of (12) can be expressed as the potential of boundary distribution of masses ν:[6]

$$\Phi_1=\int_B \nu\cdot\log r\cdot ds.\quad (13)$$

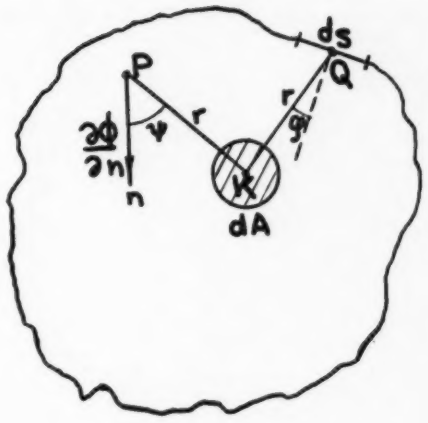

FIG. 2.

r is the distance between the point where Φ_1 is computed and a variable point on the boundary; B indicates an integration along the whole boundary.

At every point, the component F_n along a direction n of the force deriving from the potential Φ_1 is equal to:

$$F_n=\frac{\partial\Phi_1}{\partial n}=\int_B\frac{\nu}{r}\cos(n,r)ds,\quad (14)$$

if the point is not on the boundary, and is equal to

$$F_n=\int_B\frac{\nu}{r}\cos(n,r)ds+\pi\nu\quad (15)$$

if the point is on the boundary.

The general solution of (11) can be expressed as the potential of double distribution ω:[7]

$$\mu=\int_B\omega\frac{\cos\varphi}{r}ds,\quad (16)$$

in which φ is the angle between the normal at a point Q of the boundary and the radius r connecting Q to the point K where μ is computed (Fig. 2).

Expression (16), valid only if K is not on the boundary, becomes

$$\mu=\int_B\omega\frac{\cos\varphi}{r}ds+\pi\omega\quad (17)$$

if K is on the boundary.

From relations (10) and (13) the general

[5] O. D. Kellogg, *Foundations of Potential Theory* (1929), p. 150.

[6] Reference 5, p. 160.

[7] Reference 5, p. 166.

expression of Φ can be written as:

$$\Phi=\int\int_D \log r\cdot\mu\cdot dA+\int_B \log r\cdot\nu\cdot ds, \quad (18)$$

in which dA is an elemental area of the domain inside the boundary and the index D indicates the domain of integration. Equation (18) can be transformed, using Eq. (17):

$$\Phi=\int\int_D \log r\left[\int_B \frac{\omega\cdot\cos\varphi\cdot ds}{r}\right]dA + \int_B \log r\cdot\nu\cdot ds. \quad (19)$$

It is to be noted that the integration of $\pi\cdot\omega$ of (17) gives zero, as the boundary is supposed to have a volume capacity equal to zero.[8] On the boundary, the normal component of the gradient of Φ is:

$$F_n=\frac{\partial\Phi}{\partial n}=\int\int_D \frac{\cos\psi}{r}\left[\int_B \frac{\omega\cdot\cos\varphi\cdot ds}{r}\right]dA + \int_B \frac{\nu\cdot\cos\psi\cdot ds}{r}+\pi\nu. \quad (20)$$

ψ is the angle between the normal at the point where $\partial\Phi:\partial n$ is calculated and the radius r. Since Φ and $\partial\Phi:\partial n$ are known at the boundary [Eqs. (9)], Eqs. (19) and (20) define ν and ω.

Equations (19), (20) can be written as follows (indicating the variables to be integrated):

$$\Phi_P=\int\int_D \log r_{PK}\left[\int_B \frac{\omega_Q\cdot\cos(n,QK)ds_Q}{r_{QK}}\right]dA_K + \int_B \nu_Q \log r_{PQ}ds_Q, \quad (21)$$

$$\frac{\partial\Phi_P}{\partial n_P}=\int\int_D \frac{\cos(n,PK)}{r_{PK}} \times\left[\int_B \frac{\omega_Q\cdot\cos(n,QK)\cdot ds_Q}{r_{QK}}\right]dA_K + \int_B \frac{\nu_Q\cdot\cos(n,PQ)}{r_{PQ}}ds_Q+\pi\nu_P. \quad (22)$$

Notation (n, QK) designates the angle between the normal at Q and the radius QK.

[8] Reference 5, p. 330.

In order to determine the unknown functions ω and ν in function of Φ and $\partial\Phi:\partial n$ which are known on the boundary, it is necessary to make the interversion of the double and simple integrals in (21), (22).

It is necessary to show that such an operation is legitimate, i.e., that the singularities of the integrands do not make the operation impossible.

(a) If the point P is out of the domain of integration, it is obvious that the interversion is possible for domain D' interior to D, since there are no singular points in D'. In the infinitely small domain $D-D'$ (width h), the integrals can be written as follows [omitting $\log r_{PK}$, r_{PK}, $\cos(n_P, PK)$ which are all finite] (see Fig. 3):

$$\int\int_{D'-D} dA_K\int_B \frac{\omega_Q\cdot\cos(n,QK)\cdot ds_Q}{r_{QK}} \sim\int_B hds\left[\int_B \frac{\omega_Q\cdot\cos(n,QH)\cdot ds_Q}{r_{QH}}+\pi\omega\right] \leq hs\left\{\pi\omega_m+\left[\int \frac{\omega_Q\cos(n,QH)ds_Q}{r_{QH}}\right]\right\}. \quad (23)$$

In the first transformation, K is replaced by the infinitely close point H on the contour, in the second the infinitely small distance h and the arc of the boundary S are taken out of the integral and some of the terms are replaced by their maxima. It is known that

$$\int \frac{\omega_Q\cos(n,QH)ds_Q}{r_{QH}} \quad (24)$$

is bounded if ω_Q is bounded. Thus the factor multiplying h is bounded and the product tends toward zero.

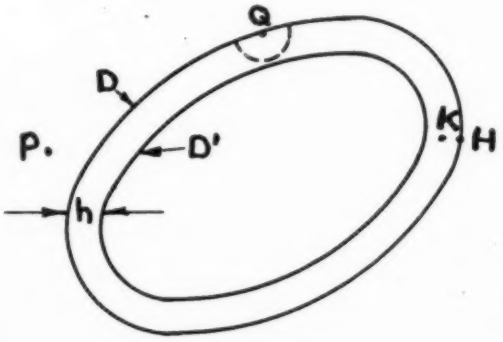

FIG. 3.

If the integral is interversed, the point Q is a fixed point in the first integration, which is made in respect to K. The integration outside a semicircular infinitely small domain surrounding Q has no singular points, and, inside that element the first integration is equivalent to:

$$\omega_Q \int\int \frac{\cos\varphi \cdot d\varphi \cdot r \cdot dr}{r} = \omega_Q \int\int \cos\varphi \cdot d\varphi \cdot dr = \omega_Q \cdot 2r \quad (25)$$

and tends toward zero with r; thus there is no singularity for the integration in respect to Q.

(b) If P is inside the domain of integration, the results under (a) being established, in addition it is necessary to verify that the integrals:

$$\int\int \log r_{PK} dA_K; \quad \int\int \frac{\cos(n, PK) \cdot dA_K}{r_{PK}} \quad (26)$$

(with K interior to the domain), have a significance. This result is a consequence of the fact that in the infinitely small domain surrounding K:

$$\int\int \log r \cdot d\varphi \cdot r \cdot dr = 2\pi \int_0^r \log r \cdot r \cdot dr, \quad (27)$$

$$\int\int \frac{d\varphi \cdot \cos\varphi \cdot r \cdot dr}{r} = 2r, \quad (28)$$

both quantities being infinitely small with r.

(c) If Q is on the boundary, the first method of integration has a significance, from results (b), since the simple integral has a significance.

The second method of integration must be examined. This reduces to the study of the integrals:

$$I_1 = \int_y \omega_Q ds \int\int \log r_{QK} \frac{\cos(n, PK)}{r_{PK}} dA_K, \quad (29)$$

$$I_2 = \int_y \omega_Q ds \int\int \frac{\cos(n, QK) \cdot \cos(n, PK)}{r_{PK} \cdot r_{QK}} dA_K, \quad (30)$$

when Q is close to P (the simple integral being effected along a small arc y), since, when Q is distant from P, there is no difficulty, as integrations around Q and P, respectively, can be effected.

When Q tends toward P, it is sufficient to consider a domain limited to a circle of radius R around P (see Fig. 4).

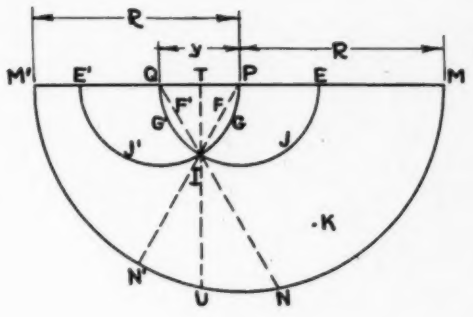

FIG. 4.

Let $r_{PK} = \rho$, it appears that integral I_1 (29) is finite when Q comes in P since:

$$I_1 < \omega_m \int ds \int\int \frac{\log\rho}{\rho} \rho d\varphi d\rho = \omega_m \int ds \cdot \pi \cdot \int \log\rho d\rho,$$

$$I_1 < \omega_m \cdot s\pi(R\log R - R). \quad (31)$$

It is possible to determine an upper limit for integral I_2 (30) when Q tends toward P.

The domain of integration (semicircle having P for center and $MM' = 2R$ for diameter) is divided into several domains having common parts and the integration is made separately in each domain.

It is obvious that

$$\frac{(MTM'UM)}{2} < (1) + (2) \quad (32)$$

in which

$$(1) = (MEJIUNM); \quad (2) = (ETIJE). \quad (32a)$$

In domain (1), $\rho = r_{PK} < r_{QK}$ and the double integral is smaller than:

$$\int\int_{(1)} \frac{dA_K}{\rho^2} = \int\int_{(1)} \frac{d\varphi d\rho}{\rho}. \quad (33)$$

Replacing domain (1) by $(MEJIN'UNM)$ of greater area and integrating:

$$\int\int_{(1)} \frac{dA_K}{\rho^2} < \frac{2\pi}{3}[R(\log R - 1) - y(\log y - 1)]. \quad (34)$$

In domain (2),

$$r_{QK} > \frac{y}{2}, \quad (35)$$

thus

$$\iint_{(2)} \frac{dA_K}{\rho^2} < \iint_{(2)} \frac{2}{y}\frac{dA_K}{\rho}. \quad (36)$$

Replacing the domain (2) by the semicircle *EQIJE:*

$$\iint_{(2)} \frac{dA_K}{\rho^2} < \frac{2\pi}{y}\int_0^y d\rho = 2\pi. \quad (37)$$

Thus, the total integral in domains (1) and (2) is smaller than:

$$2\pi + \frac{2\pi}{3}[R(\log R - 1) - y(\log y - 1)]. \quad (38)$$

That expression is finite when integrated in respect to y. Thus, the second method of integration has a significance, and the interversion of the integrations is possible. Then, the equations at the boundary can be written as follows:

$$\Phi_P = \int_B \omega_Q \cdot ds_Q \cdot \left[\iint_D \frac{\cos(n, QK)\cdot \log r_{PK}\cdot dA_K}{r_{QK}}\right] + \int_B \nu_Q \cdot \log r_{PQ} ds_Q, \quad (39)$$

$$\frac{\partial \Phi_P}{\partial n_P} = \int_B \omega_Q \cdot ds_Q \cdot \left[\iint_D \frac{\cos(n, QK)\cdot\cos(n, PK)dA_K}{r_{QK}\cdot r_{PQ}}\right] + \int_B \frac{\nu_Q \cdot \cos(n, PQ)ds_Q}{r_{PQ}} + \pi\nu_P. \quad (40)$$

Double integrals in (39), (40) are functions of the domain of integration and of the points (P, Q) and can be determined for any couple of points (P, Q). They will be designated by $\Omega(P, Q)$ and $\Psi(P, Q)$; it is to be noted that the kernel Ω is symmetrical in (P, Q) while the kernal Ψ is not symmetrical. Equations (39), (40) can be written, replacing P by V in the second equation:

$$\Phi_P = \int_B \nu_Q \cdot \log r_{PQ} \cdot ds_Q + \int_B \omega_Q \cdot \Omega(P, Q) ds_Q, \quad (41a)$$

$$\left(\frac{\partial \Phi}{\partial n}\right)_V = \int_B \nu_Q \cdot \frac{\cos(n, VQ)}{r_{VQ}} ds_Q + \pi\nu_Q + \int_B \omega_Q \Psi(V, Q) ds_Q. \quad (41b)$$

The problem is thus reduced to a system of two integral equations. The solution is obtained multiplying (41a) by an arbitrary function $M(P, V)$, integrating in respect to P, and adding the result to (41b).

Then, provided that

$$\int_B ds_P \left[M(P, V)\int \omega_Q \Omega(P, Q) ds_Q\right] + \int_B \omega_Q \Psi(V, Q) ds_Q \equiv 0 \quad (42)$$

the system (41) is transformed into

$$\left(\frac{\partial \Phi}{\partial n}\right)_V + \int_B \Phi_P M(P, V) ds_P = \int_B ds_P \left[M(P, V)\cdot \int_B \nu_Q \log r_{PQ} ds_Q\right] + \int_B \nu_Q \frac{\cos(n, VQ)}{r_{VQ}} ds_Q + \pi\nu_V. \quad (43)$$

Equation (42) can be written

$$\int_B \omega_Q ds_Q \left[\Psi(V, Q) + \int_B M(P, V)\Omega(P, Q) ds_P\right] = 0. \quad (44)$$

Since it is an identity in V, Q for any function ω, it is equivalent to

$$\Psi(V, Q) + \int_B M(P, V)\Omega(P, Q) ds_P = 0. \quad (45)$$

Integral equation (45) defines a function $M(P, V)$ in a given domain D. The function $M(P, V)$ being determined, Eq. (43) can be written as

$$\int_B \Phi_P \cdot M(P, V)\cdot ds_P + \left(\frac{\partial \Phi}{\partial n}\right)_V = \int_B \nu_Q ds_Q \left[\int_B \log r_{PQ}\cdot M(P, V) ds_P + \frac{\cos(n, VQ)}{r_{VQ}}\right] + \pi\nu_V. \quad (46)$$

Let

$$\int_B \log r_{PQ} \cdot M(P, V) ds_P + \frac{\cos(n_V, VQ)}{r_{VQ}} = \Lambda(V, Q). \quad (47)$$

Equation (46) becomes

$$\int \Phi_P \cdot M(P, V) \cdot ds_P + \left(\frac{\partial \Phi}{\partial n}\right)_V = \int \nu_Q \cdot \Lambda(V, Q) \cdot ds_Q + \pi \nu_Q. \quad (48)$$

Equation (48) is an integral equation which determines ν. Any of the Eqs. (41a), (41b) determines ω when ν is known. Thus the solution of the general problem of elasticity in two dimensions is reduced to the solution of several integral equations. The method can be summarized as follows.

PROCEDURE FOR THE SOLUTION OF TWO DIMENSIONAL PROBLEMS OF ELASTICITY IN A GIVEN DOMAIN (NO VOLUME FORCES)

(a) Calculations to be made in the domain independently of the values of the applied boundary forces.

1. Calculate the integrals:

$$\Omega(P, Q) = \int\int_D \frac{\cos(n, QK) \cdot \log r_{PK} \cdot dA_K}{r_{QK}}, \quad (49)$$

$$\Psi(P, Q) = \int\int_D \frac{\cos(n, PK) \cdot \cos(n \cdot QK) \cdot dA_K}{r_{PK} \cdot r_{QK}} \quad (50)$$

2. Solve the integral equation for the function M on the boundary:

$$\Psi(V, Q) + \int_B M(P, V) \Omega(P, Q) ds_P = 0. \quad (51)$$

3. Calculate the integral:

$$\Lambda(V, Q) = \frac{\cos(n, VQ)}{r_{VQ}} + \int_B \log r_{PQ} \cdot M(P, V) ds_P. \quad (52)$$

(b) Calculations to be made along the boundary for each loading condition, i.e., for each group of values of Φ and $\partial\Phi : \partial n$ along the boundary.

1. Solve the integral equation in ν:

$$\int_B \Phi_P \cdot M(P, Q) \cdot ds_P + \left(\frac{\partial \Phi}{\partial n}\right)_V = \int_B \nu_Q \cdot \Lambda(V, Q) ds_Q + \pi \nu_Q. \quad (53)$$

2. Solve one of the two following integral equations in ω:

$$\Phi_P - \int_B \nu_Q \cdot \log r_{PQ} \cdot ds_Q = \int_B \omega_Q \cdot \Omega(P, Q) \cdot ds_Q, \quad (54a)$$

$$\left(\frac{\partial \Phi}{\partial n}\right)_P - \int_B \frac{\nu_Q \cdot \cos(n, PQ) \cdot ds_Q}{r_{PQ}} - \pi \nu_P = \int_B \omega_Q \cdot \Psi(P, Q) ds_Q. \quad (54b)$$

3. Calculate the potential μ inside the domain D:

$$\mu_K = \int_B \frac{\omega_Q \cdot \cos(n, QK) \cdot ds_Q}{r_{QK}}. \quad (55)$$

4. The stress function Φ is defined by the relation

$$\Phi_L = \int\int_D \log r_{LK} \cdot \mu_K \cdot dA_K + \int_B \nu_Q \log r_{LQ} \cdot ds_Q. \quad (56)$$

On Van der Pol's and Related Non-Linear Differential Equations

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THE present paper is closely related to a previous one by the author.[1] First, we correct a consequential error in the said paper. Secondly, by a simple application of Parseval's formula we are able to establish directly the following fundamental result. If the equation $(d^2u/dt^2)-\epsilon F(u)(du/dt)+u=0$—generalization of Van der Pol's equation—has periodic solutions, of period close to 2π, then, under certain general conditions concerning $F(u)$, such a solution $u(t)$ is *strongly dominated* by the first term of its Fourier expansion, that is, not only $u(t)$, but also $u'(t)$, $u''(t)$, $u'''(t)$ can be approximated closely, for all t, by the said first term—simple harmonic oscillation—and its corresponding derivatives. An algebraic equation is set up for the determination of the amplitude A_1 of this harmonic oscillation, and Van der Pol's result $(A_1\sim 2)$ is confirmed. Thirdly, for Van der Pol's equation we set up expansions of $u(t)$ and its frequency ν in power series, in terms of ϵ, $0<\epsilon\ll 1$. Finally, for the same $u(t)$ and ν we obtain, for the first time, we believe, other series expansions in terms of ϵ, *for all values of* ϵ, *large and small*. The values of ν computed from the first few terms of the new series agree quite well with experimental results. The method employed is applicable to more general equations.

Corrigenda. In my paper referred to above the statement on page 43: "$\cdots|\epsilon P(t)|$, $\cdots$ in (31) and (31.1) are very small for all $t\geqq 0$" is incorrect. In fact, even a small change in the frequency of a harmonic oscillation may affect the oscillation considerably as time goes by. This simple fact had been overlooked by the author. Thus, the conclusions drawn on the basis of this statement, and similar ones applied to the non-homogeneous equation, do not hold. But what about the relation $u^2(t)+u'^2(t)\sim 4$ [Eq. (36)], which holds according to Van der Pol and which was confirmed on the differential analyzer? We show their validity below by a purely analytical treatment.

[1] J. Shohat, J. App. Phys. **14**, 40 (1943).

A. PARSEVAL FORMULA

1. Let $f(t)$ be periodic, of period T, admitting a Fourier expansion

$$f(t)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n\nu t+b_n\sin n\nu t,\quad \nu=\frac{2\pi}{T},$$
$$\begin{matrix}a_n\\ b_n\end{matrix}=\frac{2}{T}\int_0^T f(x)\begin{matrix}\cos\\ \sin\end{matrix}n\nu x\,dx;\quad n=0,1,2,\cdots. \tag{1}$$

Assuming the existence of $\int_0^T f^2(x)dx$, we have, *whether* (1) *converges or diverges*, the classical *Parseval's formula*

$$\frac{2}{T}\int_0^T f^2(x)dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}a_n^2+b_n^2. \tag{2}$$

This formula—a very powerful tool in the theory and applications of Fourier series—*is the basis for the subsequent discussion*. Consider the differential equation

$$\frac{d^2u}{dt^2}-\epsilon F(u)\frac{du}{dt}+u=0,\quad \epsilon>0. \tag{3}$$

We make here the following assumptions: (*i*) $F(u)$ is continuous for all finite u; (*ii*) $F'(u)$ exists; (*iii*) $F(u)$, $F'(u)$, hence, also $\int_0^u F(u)du$, $\int_0^u uF(u)du$, are bounded if u remains in an arbitrarily fixed finite interval [all these conditions are fulfilled if $F(u)$ is a polonomial]; (*iV*) under certain initial conditions the differential equation (3) has a continuous periodic solution, of period T, with frequency

$$\nu=\frac{2\pi}{T}.^2 \tag{4}$$

We then have the Fourier expansion

$$u(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n\nu t+b_n\sin n\nu t, \tag{5}$$

[2] Sufficiently broad conditions for the existence (and uniqueness) of periodic solutions for differential equations of the type $(d^2u/dt^2)+f(u,du/dt)(du/dt)+g(u)=0$ have been given in a recent paper by N. Levinson and C. K. Smith, Duke Math. J. **9**, 382–403 (1943).

$$\begin{matrix} a_n \\ b_n \end{matrix} = \frac{2}{T}\int_0^T u(x) \begin{matrix} \cos \\ \sin \end{matrix} n\nu x dx; \quad n=0, 1, 2, \cdots. \quad (6)$$

The expansion (5) converges for all t, since $u(t)$ and $u'(t)$ (see my paper referred to in reference 1), hence, also $u''(t)$, $u''(t)$ [by virtue of (3), where $F(u)$ satisfies the conditions stated], are continuous and bounded for all t.

Integrating (3) from 0 to T gives:

$$u'(t)\Big|_0^T - \epsilon\int_0^T F(u)du + \int_0^T udt = 0,$$

$$[u' - \epsilon\varphi(u)]_0^T + \int_0^T udt = 0, \left(\varphi(u) \equiv \int_0^u F(u)du\right),$$

whence, since u is of period T,

$$\int_0^T udt = 0.*$$

Thus, in the expansion (5) $a_0=0$, so that (5) becomes

$$u(t) = \sum_{n=1}^{\infty} A_n \sin(n\nu t + \alpha_n), \quad A_n = (a_n^2 + b_n^2)^{\frac{1}{2}}. \quad (7)$$

Furthermore,

$$u'(t) = \sum_{n=1}^{\infty} A_n n\nu \cos(n\nu t + \alpha_n), \quad (8)$$

$$u''(t) = -\sum_{n=1}^{\infty} A_n n^2\nu^2 \sin(n\nu t + \alpha_n). \quad (9)$$

(These expansions also converge for all t, by the above reasoning.) Multiply (3) by u, also by $2u'$, and integrate from 0 to T. We get:

$$[uu' - \epsilon\psi(u)]_0^T - \int_0^T u'^2 dt + \int_0^T u^2 dt$$

$$= 0\left(\psi(u) \equiv \int_0^u uF(u)du\right),$$

$$\int_0^T 2u'u''dt - 2\epsilon\int_0^T F(u)u'^2dt + \int_0^T 2uu'dt = 0,$$

$$\int_0^T u'^2dt = \int_0^T u^2dt, \quad (10)$$

$$\int_0^T F(u)u'^2dt = 0 \quad (\epsilon \neq 0). \quad (11)$$

* For the more general equation $(d^2u/dt^2) - \epsilon F(u)(du/dt) + g(u) = 0$ we get similarly $\int_0^T g(u)du = 0$, as a necessary condition for the existence of periodic solutions, so that $g(u)$ must necessarily change sign within a period.

The relation (10) is fundamental.

Thus, $F(u)$ necessarily changes sign within a period; hence, (3) cannot have periodic solutions if $F(u)$ keeps a constant sign for all u; moreover, if u_1 is the smallest numerical value of u, where $F(u)$ changes sign, then necessarily

"maximal amplitude" of $u \equiv \max_{0 \leqq t \leqq T} |u(t)| > |u_1|$.

We now turn to (10). Apply Parseval's formula (2) to either side, making use of (7) and (8):

$$\sum_{n=1}^{\infty} A_n^2 = \sum_{n=1}^{\infty} A_n^2 n^2\nu^2,$$

$$\sum_{n=1}^{\infty} A_n^2(1 - n^2\nu^2) = 0. \quad (12)$$

If $A_1^2(1-\nu^2) < 0$, then $A_n^2(1-n^2\nu^2) < 0$, for $n \geqq 2$, and (12) is impossible. If $A_1^2(1-\nu^2) = 0$, then $A_n^2(1-n^2\nu^2) \leqq 0$, $n \geqq 2$, and (12) requires $A_n^2(1-n^2\nu^2) = 0$, for $n \geqq 2$, which, barring the trivial solution $u(t) \equiv 0$, leads to the uninteresting case $A_2 = A_3 = \cdots = 0$, $u(t) = A_1 \sin(\nu t + \alpha_1)$, which correspond to $\epsilon = 0$ in (3). It follows that if $\epsilon \neq 0$ in (3), the first term in (12) is necessarily positive, so that

$$\nu < 1, \quad T > 2\pi. \quad (13)$$

2. $0 < \epsilon \ll 1$. For $\epsilon = 0$, all solutions of (3) have period 2π, i.e., $\nu = 1$. Assume that for $0 < \epsilon \ll 1$, the period is close to 2π, that is, $\nu \sim 1$ (this holds under certain general conditions). How does the periodic solution behave? This question is of particular significance, since it was shown (Van der Pol, Liénard, Levinson and Smith) that, under certain general conditions, the periodic solution is unique (disregarding a time-shift) and that all other solutions approach it asymptotically, as $t \to \infty$. Take ϵ so small that

$$\nu \equiv \nu(\epsilon) > \frac{\sqrt{2}}{2}. \quad (14)*$$

Then

$$n^2\nu^2 - 1 > 0, \text{ for } n \geqq 2, \quad (15)$$

and (12) yields

$$A_1^2(1-\nu^2) = A_2^2(4\nu^2 - 1) + A_3^2(9\nu^2 - 1) + \cdots, \quad (16)$$

* See below, Part B, where we treat Van der Pol's equation by means of power series in ϵ and derive an estimate of $\nu(\epsilon)$.

where all terms are positive, whence

$$A_n < \frac{A_1(1-\nu^2)^{\frac{1}{2}}}{(n^2\nu^2-1)^{\frac{1}{2}}}, \quad n \geqq 2, \tag{17}$$

which shows the magnitude of the coefficients $A_2, A_3, \cdots$ in the Fourier expansion (7), compared to A_1. More precisely, write

$$u(t) = A_1 \sin(n\nu t+\alpha_1) + Q(t,\epsilon),$$

$$Q(t,\epsilon) = \sum_{n=2}^{\infty} A_n \sin(n\nu t+\alpha_n) = \sum_{n=2}^{\infty} A_n(n^2\nu^2-1)^{\frac{1}{2}} \cdot \frac{\sin(n\nu t+\alpha_n)}{(n^2\nu^2-1)^{\frac{1}{2}}}.$$

Apply Cauchy inequality $((\Sigma a_n b_n)^2 \leqq \Sigma a_n^2 \Sigma b_n^2)$ to the right-most term above. This gives:

$$|Q(t,\epsilon)| \leqq \left(\sum_{n=2}^{\infty} A_n^2(n^2\nu^2-1) \cdot \sum_{n=2}^{\infty} \frac{1}{n^2\nu^2-1}\right)^{\frac{1}{2}},$$

for all t, and since, by (14),

$$n^2 - \frac{1}{\nu^2} > n^2 - 2 > (n-1)^2, \qquad n \geqq 2,$$

we get finally, making use of (16) and recalling that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}:$$

$$u(t) = A_1 \sin(\nu t+\alpha_1) + Q(t,\epsilon),$$

$$|Q(t,\epsilon)| < \frac{\pi}{\sqrt{6}} \frac{A_1}{\nu}(1-\nu^2)^{\frac{1}{2}} < 2A_1(1-\nu^2)^{\frac{1}{2}} = O((1-\nu^2)^{\frac{1}{2}}).* \tag{18}$$

This holds for all t, under assumption (14). Rewrite (18) as

$$u(t) = A \sin\gamma + 2A_1(1-\nu^2)^{\frac{1}{2}} \cdot \theta(t,\epsilon), \quad (\gamma = \nu t+\alpha_1)$$
$$|\theta(t,\epsilon)| < 1 \text{ for all } t. \tag{19}$$

$u(t)$ thus differs but little from the simple harmonic oscillation $A_1 \sin(\nu t+\alpha_1)$ for all t. What about its derivatives?

3. By means of (19) and (2) we get readily:

$$\frac{T}{2}\sum_{n=1}^{\infty} A_n^2 = \int_0^T u^2 dt = \int_0^T u'^2 dt = \frac{TA_1^2}{2}(1+\eta_1), \quad \eta_1 = O((1-\nu^2)^{\frac{1}{2}}), \tag{20}$$

$$\sum_{n=2}^{\infty} A_n^2 n^2\nu^2 = O((1-\nu^2)^{\frac{1}{2}}). \tag{21}$$

[It will be shown below, on the basis of (18), that $A_1 \sim 2$.]

$$\sum_{n=2}^{\infty} A_n^2 = A_1^2\eta_1 = O((1-\nu^2)^{\frac{1}{2}}). \tag{22}$$

Again from (3)

$$\int_0^T u''^2 dt - \epsilon\int_0^T F(u)u'u''dt + \int_0^T uu''dt = 0. \tag{23}$$

By (2), (9), and (19),

$$I \equiv \int_0^T u''^2 dt = \frac{T}{2} \cdot \sum_{n=1}^{\infty} A_n^2 n^4\nu^4, \tag{24}$$

$$|u(t)| \leqq A_1[1+2(1-\nu^2)^{\frac{1}{2}}] < 3A_1, \text{ for all } t. \tag{25}$$

Now, by assumption, (25) implies $|F(u)| <$ a certain constant C, for all t. Apply Schwarz inequality

$$\left(\left(\int_a^b f_1(x)f_2(x)dx\right)^2 \leqq \int_a^b f_1^2(x)dx \cdot \int_a^b f_2^2(x)dx\right)$$

to the second and third integral in (23). This gives, by (22):

$$\left|\int_0^T F(u)u'u''dt\right| \leqq C\left(\int_0^T u'^2 dt \cdot \int_0^T u''^2 dt\right)^{\frac{1}{2}} < C[A_1^2 \cdot T/2 \cdot (1+\eta_1)]^{\frac{1}{2}} \cdot I^{\frac{1}{2}},$$

$$\left|\int_0^T uu''dt\right| \leqq \left(\int_0^T u^2 dt \cdot \int_0^T u''^2 dt\right)^{\frac{1}{2}} \leqq [A_1^2(T/2)(1+\eta_1)]^{\frac{1}{2}} \cdot I^{\frac{1}{2}}.$$

Equation (23) now becomes

$$I - [A_1^2 \cdot T/2 \cdot (1+\eta_1)]^{\frac{1}{2}}(-\epsilon\theta_1 C + \theta_2) \cdot I^{\frac{1}{2}} = 0,$$

$\theta_{1,2} = \text{const.}$, $|\theta_{1,2}| \leqq 1$, whence

$$\sum_{n=2}^{\infty} A_n^2 n^4\nu^4 \leqq A_1^2(1-\nu^4+O(\delta)) = O(\delta), \tag{26}$$

$\delta = \epsilon$ or $(1-\nu^2)^{\frac{1}{2}}$, whichever is of smaller order of magnitude. Thus, $A_2, A_3, \cdots$, are even smaller than given by (21). By virtue of (26), we now get:

$$u'(t) = A_1\nu \cos(\nu t+\alpha_1) + Q'(t,\epsilon) \tag{27}$$

$$Q'(t,\epsilon) = \sum_{n=2}^{\infty} A_n n\nu \cos(n\nu t+\alpha_n) = O(\delta^{\frac{1}{2}}) \text{ for all } t.$$

* The now customary notation $x = O(\alpha)$ means: x is of the same order of magnitude as α; for example, $(2n^3-3)^{-\frac{1}{2}} = O(n^{-\frac{3}{2}})$, as $n \to \infty$.

The estimate of $Q'(t, \epsilon)$ is derived by a reasoning quite similar to that employed above in estimating $Q(t, \epsilon)$. Combining (18) and (27), we get:

$$u^2(t)+u'^2(t)\sim A_1^2, \text{ for all } t, \text{ error}=O(\delta^{\frac{1}{2}}). \quad (28)$$

The approximations to $u''(t)$, $u'''(t)$ are now furnished by the differential equation (3) itself, namely:

$$\begin{aligned} u''(t) &= -A_1\nu^2 \sin(\nu t+\alpha_1)+Q''(t,\epsilon), \\ &\qquad |Q''(t,\epsilon)| = O(\delta) \\ u'''(t) &= -A_1\nu^3 \cos(\nu t+\alpha_1)+Q'''(t,\epsilon), \text{ for all } t. \\ &\qquad |Q'''(t,\epsilon)| \ll 1. \end{aligned} \quad (29)$$

Thus, *if $\epsilon \ll 1$, the periodic solution $u(t)$ of the differential equation (3) and its first three derivatives can be approximated, for all t, by the simple harmonic oscillation $A_1 \sin(\nu t+\alpha_1)$, which represents the first term in the Fourier expansion of $u(t)$, and its corresponding derivatives.*

4. We proceed to evaluate A_1. Here we make use of (11). Assume $F(u)$ is a polynomial:

$$F(u)=\alpha_0+\alpha_1 u+\cdots+\alpha_p u^p. \quad (30)$$

By (18) and (27), for any fixed positive integer s and for all t:

$$u^s = A_1^s \sin^s\gamma+\eta_s, \quad \eta_s \equiv \eta_s(t,\epsilon)=O((1-\nu^2)^{\frac{1}{2}}),$$

$$u'^2 = A_1^2\cos^2\gamma+O(\delta^{\frac{1}{2}}).$$

Substitute this into (11):

$$0=\sum_{s=0}^{p}\int_0^T \alpha_s A_1^{s+2}\sin^s\gamma\cos^2\gamma\,dt+O(\delta^{\frac{1}{2}}). \quad (31)$$

But

$$\int_0^T \sin^{2i+1}\gamma\cos^2\gamma\,dt = -\int_0^T (1-\cos^2\gamma)^i\cos^2\gamma\,d(\cos\gamma)=0.$$

We conclude that *odd powers of u in $F(u)$ have no influence on A_1, within our approximation*, so that we may keep in $F(u)$ even powers only. Returning to (31), where we may set $\alpha_1=\alpha_3=\cdots=0$, and using the simple formula

$$\int_0^T \sin^{2i}\gamma\cos^2\gamma\,dt=\frac{T}{2i+1}\cdot\frac{1\cdot3\cdots(2i-1)}{2\cdot4\cdot6\cdots2i}, \quad i\geqq 1,$$

we get the desired equation for evaluating A_1:

$$\alpha_0+\sum_{i=1}^{m}\alpha_{2i}\cdot\frac{1\cdot3\cdots(2i-1)}{2\cdot4\cdot6\cdots2i}\cdot\frac{z^i}{i+1}=0, \quad z=A_1^2. \quad (32)$$

Note that in (32) the coefficients are independent of ϵ and T. The same equation was obtained in an entirely different way by Liénard.[3] We conclude that *A_1—the square of the amplitude of the fundamental harmonic of $u(t)$—is close to a root of Eq. (32)*. Hence, *a necessary condition for the existence of periodic solutions for Eq. (3)*, [$F(u)$ having the above properties] is: *(32) possesses at least one non-negative root.*

As to the phase α_1 in (18), it is of no special significance, for we can replace t in $u(t)$ by $t+$arbitrary constant, since time does not enter explicitly in (3). We may set, if we wish, $\alpha_1=0$.

1. Illustration. $F(u)=1+au-u^{2n}$. Here (3) possesses a unique periodic solution (cf. the paper quoted in reference 2). Equation (32) becomes

$$1-\frac{1\cdot3\cdots(2m-1)}{2\cdot4\cdots2m}\cdot\frac{z^m}{m+1}=0,$$

whence

$$A_1\sim\left[\frac{2\cdot4\cdot6\cdots2m}{1\cdot3\cdot5\cdots(2m-1)}\cdot(m+1)\right]^{1/2m}. \quad (33)$$

The cases $m=1$, 2 yield, respectively, $A_1\sim 2$, $A_1\sim 8^{\frac{1}{2}}$. Both results agree with those previously derived by the author* ($a=0$, $m=1$ corresponds to Van der Pol's case).

Remark. From the classical results of Bendixson[4] it follows that, for $\epsilon>0$, any solution of Van der Pol's equation approaches indefinitely, for $t\to\infty$, a stationary state represented by its unique periodic solution, which, in turn, if $0<\epsilon\ll 1$, can be closely approximated, for all t, by the simple harmonic oscillation $2\sin(\nu t+\alpha_1)$, as shown above. This is the fundamental result of Van der Pol. Thus, our periodic solution is stable. Its behavior and analytic representation are of fundamental importance for all solutions.

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[3] A. Liénard, Rev. Gen. de l'Elec. **23**, 901–946 (1928); Cf. also: N. Kryloff and N. Bogoliouboff, *Introduction to Non-Linear Mechanics*, free translation by S. Lefschetz (Princeton University Press, 1943).

* The estimates (18) and (27) which hold for all t justify the method employed by the author to evaluate A_1 in his previous paper.

[4] L. Bendixson, Acta Math. **24**, 1–88 (1901). Cf. also Levinson and Smith, and Liénard, references 2 and 3.

B. $0<\epsilon\ll 1$. APPLICATION OF POWER SERIES

We exhibit the method, essentially due to Lindstedt,[5] on Van der Pol's equation.

$$\frac{d^2u}{dt^2}-\epsilon(1-u^2)\frac{du}{dt}+u=0. \tag{34}$$

It applies to the more general equation (3), $F(u)$ being a polynomial.

5. We start with the expansion of the periodic solution $u(t)$ and of the frequency $\nu=2\pi/T$ in power series of the parameter ϵ. However, a straightforward substitution into (34) of a power series in ϵ for $u(t)$ is of no avail for it brings out so called "secular terms," like $t\cdot\frac{\cos}{\sin} nt$. We proceed as follows. Introduce a new independent variable

$$x=2\pi t/T, \tag{35}$$

where T is the original period of $u(t)$, so that u becomes a function of x, *of period* 2π. Equation (35) transforms (34) into

$$\nu^2u''-\epsilon\nu(1-u^2)u'+u=0, \tag{36}$$

where primes denote differentiation with respect to x. Here we substitute

$$\nu=\gamma_0+\gamma_1\epsilon+\gamma_2\epsilon^2+\cdots, \quad \gamma_i=\text{const.}, \tag{37}$$

$$u=\psi_0(x)+\psi_1(x)\epsilon+\psi_2(x)\epsilon^2+\cdots \tag{38}$$

$$\frac{du}{dx}\equiv u'=\psi_0'(x)+\psi_1'(x)\epsilon+\psi_2'(x)\epsilon^2+\cdots,$$

where the ψ's are periodic functions, of period 2π, which we take as represented by Fourier series. Since t does not enter explicitly into (34), we may fix the t origin, $t=0$, so that

$$\frac{du(t)}{dt}=0 \text{ for } t=0, \quad u(0)=\max u(t), \quad 0\leqq t\leqq T. \tag{39}$$

Then in (36) $u'(x)=0$ and $u(x)$ is maximum for $x=0$. Letting

$$u(0)=\alpha_0+\alpha_1\epsilon+\alpha_2\epsilon^2+\cdots, \tag{40}$$

it results:

$$\psi_i(0)=\alpha_i, \quad \psi_i'(0)=0; \quad i=0, 1, 2, \cdots. \tag{41}$$

[5] Lindstedt, Mém. Acad. Sci. St.-Pétersbourg **31** (1883); cf. also Poincaré, *Les méthodes nouvelles de la Mécanique Céleste* (Paris, 1893), Vol. 11, where Linstedt's series, "interesting and useful," are elaborated in great detail.

Since (34) does not change if we replace t by $-t$ and ϵ by $-\epsilon$, hence, x by $-x$ and ϵ by $-\epsilon$ in (36). This suggests the determination of the $\psi-s$ so as to have, for $i\geqq 0$,

$$\psi_{2i}(-x)\equiv\psi_{2i}(x), \quad \psi_{2i+1}(-x)\equiv-\psi_{2i+1}(x), \tag{42}$$

$$\psi_{2i}(x)=\textstyle\sum_l a_{il}\cos lx,$$

$$\psi_{2i+1}(x)=\textstyle\sum_l b_{il}\sin lx,$$

$$\textstyle\sum_l lb_{il}=0, \tag{43}$$

provided this leads to $\nu(-\epsilon)\equiv\nu(\epsilon)$ [see formula (55.1) below]. Then $\alpha_1=\alpha_3=\cdots=\gamma_1=\gamma_3=\cdots=0$,

$$u(0)=\alpha_0+\alpha_2\epsilon^2+\alpha_4\epsilon^4+\cdots,$$

$$\nu=\gamma_0+\gamma_2\epsilon^2+\gamma_4\epsilon^4+\cdots. \tag{44}$$

[Part C obviates the necessity of showing that all $\psi-s$ can actually be determined in this way.] This greatly facilitates our computations.

We now substitute (38) and (44) into (36). Annulling various powers of ϵ yields a sequence of differential equations for the successive determination of $\psi_0, \psi_1, \psi_2, \cdots$, all of period 2π, subject to the initial conditions

$$\psi_{2i}(0)=0, \quad \psi_{2i+1}'(0)=0; \quad i=1, 2, 3, \cdots \tag{45}$$

[the remaining initial conditions (41) are satisfied automatically by (44)]. Here are the differential equations for $\psi_0, \psi_1, \psi_2, \psi_3$:

$$\gamma_0\psi_0''+\psi_0=0, \tag{46}$$

$$\psi_1''+\psi_1=(1-\psi_0^2)\psi_0', \tag{47}$$

$$\psi_2''+\psi_2=-2\gamma_2\psi_0''+(1-\psi_0^2)\psi_1'-2\psi_0\psi_0'\psi_1, \tag{48}$$

$$\psi_3''+\psi_3=-2\gamma_2\psi_1''+\gamma_2(1-\psi_0^2)\psi_0'-2\psi_0\psi_1\psi_1'+(1-\psi_0^2)\psi_2'-(\psi_1^2+2\psi_2\psi_0)\psi_0'. \tag{49}$$

The left-hand member in each of these equations is of the form $\ddot{y}''+y=0$, whose general solution is $A\cos x+B\sin x$. *Thus, the constants* α_{2i}, γ_{2i} *will be determined by the disappearance from the right-hand members in Eqs. (46)–(49) of terms of the type const.* $x\frac{\cos}{\sin} x$, for the presence of such terms brings out secular terms. This is the main feature of Lindstedt's method. We get first

$$\gamma_0=1, \quad \psi_0=\alpha_0\cos x,$$

$$\psi_1''+\psi_1=\left(\frac{\alpha_0^3}{4}-\alpha_0\right)\sin x+\frac{\alpha_0^3}{4}\sin 3x. \tag{50}$$

Hence,

$$\frac{\alpha_0^3}{4}-\alpha_0=0, \quad \alpha_0=2 \text{ (for } \alpha_0\sim u_{\max}),$$
$$\psi_0(x)=2\cos x. \tag{51}$$

Equation (50) now becomes

$$\psi_1''(x)+\psi_1(x)=2\sin 3x,$$

and yields, by virtue of (44),

$$\psi_1(x)=\tfrac{3}{4}\sin x-\tfrac{1}{4}\sin 3x=\sin^3 x. \tag{52}$$

A somewhat simpler method for determining ψ_1 and other ψ's is as follows:

Integrate (47) from 0 to x, writing $\psi_1(x)=\Sigma_l b_l \sin lx$, with $\Sigma_l lb_l=0$. This gives:

$$\sum_l b_l\left(l-\frac{1}{l}\right)\cos lx+\sum_l\frac{b_l}{l}$$
$$=\left(2\cos x-\frac{8\cos^3 x}{3}\right)\Bigg|_0^x=\tfrac{2}{3}-\tfrac{2}{3}\cos 3x,$$

$l=1, 2, 3; \quad b_2=0, \quad b_3=-\tfrac{1}{4}, \quad b_1=-3b_3=\tfrac{3}{4}.$

The same method, applied to (48), yields for γ_2 and $\psi_2(x)=\Sigma_l a_l\cos lx$:

$$\gamma_2=-\tfrac{1}{16}, \tag{53}$$
$$\psi_2(x)=A\cos x+\tfrac{3}{16}\cos 3x-\tfrac{5}{96}\cos 5x. \tag{54}$$

In order to find A and $\psi_3(x)$, apply the same method to (49), rewritten as

$$\psi_3''+\psi_3=-2\gamma_2\psi_1''+\gamma_2(\psi_0-\psi_0^3/3)'$$
$$+[(1-\psi_0^2)\psi_2-\psi_0\psi_1^2]'.$$

We find $A=-\tfrac{1}{8}$. Thus,

$$u(x)=2\cos x+\epsilon\sin^3 x$$
$$+\epsilon^2(-\tfrac{1}{8}\cos x+\tfrac{3}{16}\cos 3x-\tfrac{5}{96}\cos 5x)+\cdots,$$
$$2+\sqrt{2}\epsilon+\cdots\geqq u_{\max}=u(0)=2+\tfrac{1}{96}\epsilon^2+\cdots, \tag{55}$$
$$\nu=1-\tfrac{1}{16}\epsilon^2+\cdots, \quad T=\frac{2\pi}{\nu}=2\pi(1+\tfrac{1}{16}\epsilon^2+\cdots).$$

[The left-hand inequality for $u_{\max}$ follows from (25).] We may check our computation by the following relation derived from (36), (51), and (54):

$$\nu^2=-\frac{u(0)}{u''(0)}=\frac{2+(A+1\tfrac{3}{96})\epsilon^2+\cdots}{2+(A+3\tfrac{7}{96})\epsilon^2+\cdots}$$
$$=1-\tfrac{1}{8}\epsilon^2+\cdots. \tag{55.1}$$

We conclude that

$$(1-\nu^2)^{\frac{1}{2}}=O(\epsilon), \tag{56}$$

and this enables us to make more precise the estimates for $u(t)$, $u'(t)$, $\cdots$, arrived at in Part A. Namely, we can replace in formulae (20), (21), etc., $O((1-\nu^2)^{\frac{1}{2}})$, $O(\delta)$, $O(\delta^{\frac{1}{2}})$ by $O(\epsilon)$, $O(\epsilon)$, $O(\epsilon^{\frac{1}{2}})$, respectively.

6. *The case $\epsilon\gg 1$.* Many of the foregoing results hold, like formulae (7)–(11). Thus, we again have (12), with the resulting inequality

$$\nu<1, \quad T>2\pi.$$

However, here ν becomes very small, as ϵ becomes very large. In fact, Van der Pol and Liénard have shown that

$$T\sim 1.6\cdots\epsilon, \quad \nu\epsilon\sim 3.89\cdots \quad (\epsilon\to\infty). \tag{57}$$

It follows that *in (12) several terms at the beginning are positive*. Assume, for example,

$$\frac{1}{(k+1)^2}<\nu^2\leqq\frac{1}{k^2}.$$

Then we write, in place of (16),

$$A_1^2(1-\nu^2)+A_2^2(1-4\nu^2)+\cdots+A_k^2(1-k^2\nu^2)$$
$$=\sum_{n=k+1}^{\infty}A_n^2(n^2\nu^2-1),$$

where all terms are non-negative. This suggests, as was stated by Van der Pol, that in the Fourier expansion (7) for $u(t)$ *several terms from the beginning dominate* (which may account for the complicated graphical picture, as given by Van der Pol), and the analysis of Part A becomes greatly complicated. The power series expansions, as developed above, obviously cannot be applied here. It would be natural to attempt expansions in negative powers of ϵ, but the very first equation for $\psi_0(x)$ [see (46)–(49)] yields a non-periodic complicated function, and this bars the determination of ψ_1, ψ_2, $\cdots$, as periodic functions. This difficulty has been overcome in the last part, C, by means of a new type of series for $u(x)$ and ν.

C. SERIES EXPANSIONS FOR u AND ν, FOR ALL VALUES OF ϵ

7. The method which follows is applicable to any equation of the type (3), $F(u)$ being a polynomial, where (Liénard) $\nu\sim 1/\epsilon$ for large ϵ, [see (57)]. We apply it in detail to Van der Pol's equation (34), it being the most interesting

equation of this type, which played an important part in applied sciences, in developing the theory of relaxation oscillations, and indeed, the theory of non-linear differential equations of physics and engineering.

Again we transform (34) into (36) by means of the substitution (35). The main feature of our method is to consider the equation

$$\nu^2 u'' - \nu\epsilon\rho(1-x^2)u' + u = 0, \tag{58}$$

where ρ is a new parameter in terms of which we expand $u(x)$ and ν; ultimately ρ is replaced by unity. As before, we set

$$u(x) = \sum_{i=0}^{\infty} \psi_i(x)\rho^i, \tag{59}$$

ψ_i periodic, of period 2π,

$$u'(0) = 0, \quad u(0) = u_{\max};$$

$$\psi_i'(0) = 0, \quad i = 0, 1, 2, \cdots.$$

Guided by (57), we now set

$$\epsilon = \frac{1}{1+h-\rho},$$

$$\nu = (1+h-\rho)(\gamma_0 + \gamma_1\rho + \gamma_2\rho^2 + \cdots), \tag{60}$$

where h and γ_i are constants. We note that $\rho = 1$ implies $\epsilon = 1/h$, $\epsilon \ll 1$ for h very large, $\epsilon \gg 1$ for h very small, and we may anticipate that *our series will serve for all values of ϵ, large and small.*

Now substitute (59) and (60) into (58), annul various powers of ρ, and obtain a system of differential equations for ψ_0, ψ_1, $\cdots$ [see (46)–(49)], the first of which is

$$k^2\gamma_0\psi_0'' + \psi_0 = 0, \quad k = 1+h. \tag{61}$$

We get, since $\psi_0(x)$ is of period 2π and $\psi_0'(0) = 0$,

$$\gamma_0 = \frac{1}{k} = \frac{1}{1+h}, \quad \psi_0 = \alpha_0 \cos x. \tag{62}$$

The equation for ψ_1 now becomes:

$$\psi_1'' + \psi_1 = -2(k\gamma_1 - \gamma_0)\psi_0'' + \gamma_0(1-\psi_0^2)\psi_0'.$$

As in Part B, we seek to avoid secular terms, and the calculations are quite similar. Setting $\rho = 1$ yields:

$$k = 1+h = \frac{\epsilon+1}{\epsilon}, \quad \gamma_0 = \frac{\epsilon}{1+\epsilon}, \quad \gamma_1 = \left(\frac{\epsilon}{1+\epsilon}\right)^2.$$

$$\psi_0(x) = 2\cos x, \quad \psi_1(x) = \frac{\epsilon}{1+\epsilon}\sin^3 x.$$

The equation for ψ_2 now becomes:

$$\psi_2'' + \psi_2 = -2(k\gamma_2 - \gamma_1)\psi_0'' - 2\gamma_0\psi_0\psi_1\psi_0' + \gamma_1(1-\psi_0^2)\psi_0' + \gamma_0(1-\psi_0^2)\psi_1'$$

and yields $\psi_2(x)$ in a similar manner. Thus,

$$u(x) = 2\cos x + \frac{\epsilon}{1+\epsilon}\sin^3 x + \frac{\epsilon^2}{(1+\epsilon)^2} \times [-\tfrac{1}{8}\cos x + \tfrac{3}{16}\cos 3x - \tfrac{3}{96}\cos 5x + \sin^3 x] + \cdots, \tag{63}$$

$$\nu = \frac{1}{1+\epsilon} + \frac{\epsilon}{(1+\epsilon)^2} + \frac{15}{16}\frac{\epsilon^2}{(1+\epsilon)^3} + \frac{13}{16}\frac{\epsilon^3}{(1+\epsilon)^4} + \cdots. \tag{64}$$

It is interesting to note that assuming $\epsilon \ll 1$ and expanding in (63) and (64) each coefficient in powers of ϵ, we obtain the same series for u and ν as in Part B.

Table I contains, in two columns, values of ν

TABLE I.

ϵ	ν (from series (64))	ν (according to Van der Pol)
0.1	0.998	1.
0.33	.98	0.99
1	.93	0.90
2	.77	0.78
5.6	.43	0.38(?)
8	.35	0.39
10	.30	0.31
11.4	.25	0.24
99	.037	—

for various values of ϵ, calculated from the first four terms in (64), also from the experimental curves obtained by Van der Pol (for the latter I am indebted to my Moore School colleague Professor C. N. Weygandt, Jr.). The agreement between the two columns is quite good. The first four terms in our series (64) give further

$$\nu\epsilon \sim 3.75 \quad (\epsilon \to \infty),$$

which again agrees well with Liénard's theoretical results: $\nu\epsilon \sim 3.89\cdots$. Moreover, the same terms in (64) gives $d\nu/d\epsilon < 0$, which makes plausible the following answer to a question raised by Liénard. As the parameter ϵ in Van der Pol's equation increases from 0 to ∞, the frequency ν of its periodic solution decreases from 1 to 0, hence, the period T increases from 2π to ∞.

The foregoing method is offered for the purpose of computations. I wish to thank my pupil, Mr. J. D. Chapline, Jr., for checking the above numerical results for $u(x)$ and ν.

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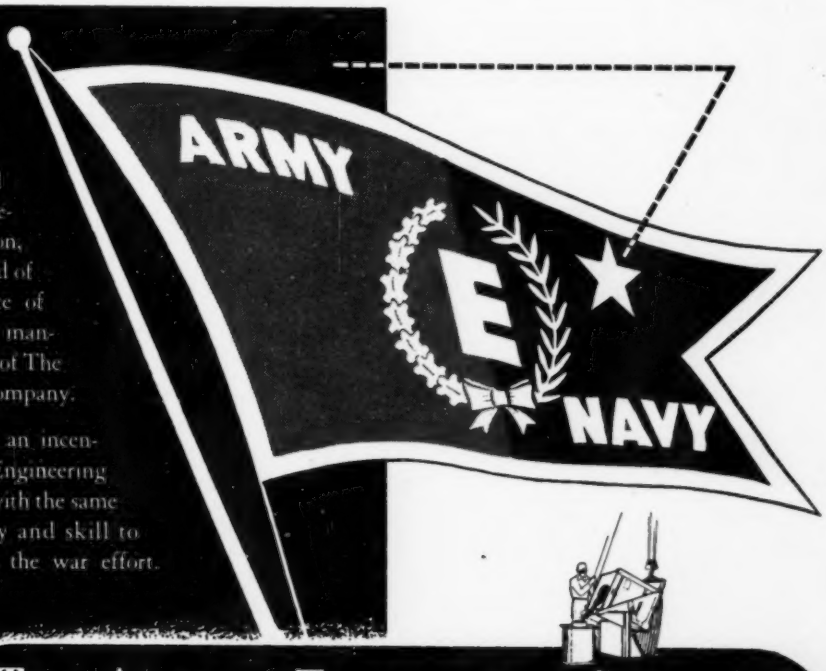
The star will serve as an incentive to The Arnold Engineering Company to continue with the same high devotion, energy and skill to turn out products for the war effort.



THE ARNOLD ENGINEERING COMPANY

147 EAST ONTARIO STREET, CHICAGO 11, ILLINOIS

Specialists in the Manufacture of ALNICO PERMANENT MAGNETS

Recent Applications of Physics

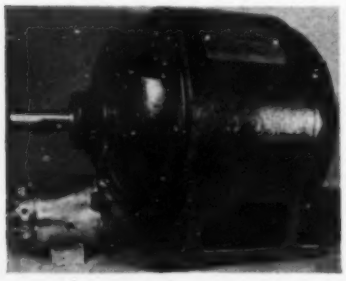
Prepared by CLARK GOODMAN, Associate Editor

Material to be included in this section should be submitted to Dr. Clark Goodman, Massachusetts Institute of Technology, Cambridge 39, Massachusetts.

Electrostatic Dust Precipitator The cleaning action upon gases of a corona discharge has been known for more than a century. The Cottrell precipitator has long been recognized as the outstanding method of removing the very fine particles from gases, especially at elevated temperatures. The more general use of this technique for the removal of undesirable mist, dust, and pollen has been delayed for a number of reasons. The cost and maintenance are high as compared to other types of cleaning equipment. Considerable space and appreciable currents at d.c. voltages of from 30,000 to 100,000 are required. In addition, this type of precipitator generates so much ozone that the clean air, although free of dust, is too irritating to the nose and throat to be used for ventilation. Recently Westinghouse engineers have been able to overcome most of these objections in an electrostatic unit known as the Precipitron. By careful design a compact unit small enough to be placed in an ordinary window has been developed. The generation of ozone by a given discharge current depends on the geometry of the electrodes. By selecting a favorable design of electrode the generation of ozone per unit of current has been reduced to the point where its concentration in the air is comparable with that found in outdoor air in bright sunlight.

The Precipitron has been utilized to remove oil mist from the atmosphere surrounding high speed grinders cooled by cutting oil. The expensive oil is recovered for re-use. Precipitrons have also been used to remove dust from the large volume of warm air required for dehydration of milk. The removal of dust from blast-furnace gases is sufficiently complete that these gases can be used as fuel for engines. Persons suffering from hay fever and certain forms of asthma are greatly relieved by the electrostatic removal of pollens from the air. Among the more promising postwar applications of electrostatic precipitation are the removal of road dust at the air intake of automobile engines and as a much needed supplement to present railway air-conditioning units.

Mighty Midget Motor According to General Electric engineers, the surface speed required to produce a high grade finish economically by grinding should be about the same whether grinding a large or a small hole. For 1-inch diameter or larger, medium-size wheels rotating at moderate speeds give the requisite surface speed. For small holes, many less than $\frac{1}{4}$ inch in diameter, a tiny grinding wheel rotating at a high speed is needed. A new G.E. motor has been developed for this purpose. It weighs but seven pounds, yet is rated 3 horsepower. With normal voltage applied at 2000 cycles, the midget motor reaches its full speed of 120,000 r.p.m. in less than a second. Tests have shown that these motors can deliver the rated power at this speed for more than eight hours of continuous operation. The amount of material used in the motor is so small, with a consequent reduction in radiating surface, that water cooling is necessary, one-half a gallon per minute being used. For contrast, the small motor is compared in the accompanying photograph with the standard 3-hp, 1800-r.p.m. motor which weighs 105 pounds.



Vectographic Views Modern three-dimensional photography has antiquated the old stereoscope that provided so much parlor entertainment in the good old days. No longer is only one person at a time permitted to view a picture in three dimensions. Instead, an entire class or a large audience can see the same three-dimensional photograph or drawing when projected on a screen and viewed through special polarizing spectacles. The techniques that make this possible were developed by the Polaroid Corporation. The products are called vectographs. They can be equally well used as prints or lantern slides. Both are filling urgent war needs in teaching military navigation and in aerial reconnaissance. In the training program a textbook containing vectographic prints is used in place of the conventional two-dimensional diagrams which attempt to represent spatial relationships. Celestial navigation vectograph slides, projected by standard projectors, are so strikingly realistic that students feel they are looking at precise wire models of the heavens with relative positions of the stars and the earth immediately apparent. By revealing land details and the positions of enemy installations, vectographs enable key men in combat to know the lay of the land before they engage in offensive action against objective areas. Vectographs also offer the usual advantages of three-dimensional studies of bombing targets both before and after the attack.

Vectographs are prepared from the two views or *stereograms*, required for all three-dimensional pictures, by a simple imbibition-transfer process. Prints of the stereoscopic negatives—one normal and one reversed—are made on Eastman Wash-Off Relief Film. After development the picture images appear in relief on the gelatin surfaces. These reliefs, while still wet, are hinged together with their emulsion sides face to face and soaked briefly in a printing solution. A sheet of vectograph film, resembling a clear acetate sheet, is placed between them, and the sandwich is run through a wringer. The reliefs are then peeled from the vectograph film, which, as a result of these operations, has stereoscopic prints of the object on its two surfaces. Polarizing spectacles, in conjunction with the polarizing element in the vectograph print, permit the two eyes to see mutually exclusive images which the brain interprets as the normal seeing condition. As many as 2000 vectographs can be made from a single set of relief films. The dried vectograph film may be mounted directly as a lantern slide or used as a print by painting the front surface with clear lacquer and the back surface with aluminum lacquer.